

Size shrinking of composite bosons for increasing density in the BCS to Bose-Einstein crossover

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Abstract. We consider a system of fermions in the continuum case at zero temperature, in the strong-coupling limit of a short-range attraction when composite bosons form as bound-fermion pairs. We examine the density dependence of the size of the composite bosons at leading order in the density (“dilute limit”), and show on general physical grounds that this size should *decrease* with increasing density, both in three and two dimensions. We then compare with the analytic zero-temperature mean-field solution, which indeed exhibits the size shrinking of the composite bosons both in three and two dimensions. We argue, nonetheless, that the two-dimensional mean-field solution is not consistent with our general result in the “dilute limit”, to the extent that mean field treats the scattering between composite bosons in the Born approximation which is known to break down at low energy in two dimensions.

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1 Introduction

The BCS to Bose-Einstein (BE) crossover can be regarded as an evolution from large overlapping Cooper pairs (BCS limit) to small nonoverlapping (composite) bosons (BE limit). At zero temperature, this crossover has thus been characterized in the continuum case in terms of the correlation length ξ_{pair} for pairs of opposite-spin fermions (in units of k_F^{-1} , where k_F is the Fermi wave vector) [1].

It was found that, by reducing the strength of the fermionic attraction at fixed density, ξ_{pair} increases monotonically from its strong-coupling (BE) limit (equal to the bound-state radius of the associated two-body problem) to the Pippard value in the weak-coupling (BCS) limit.

It was emphasized recently that, as the particle density increases for fixed value of the potential strength, the system also evolves from the BE limit toward the crossover region, eventually reaching the BCS limit (provided certain conditions on the fermionic attraction are fulfilled) [2].

In this context, one would naively expect ξ_{pair} to increase monotonically, too, when evolving from the BE limit toward the crossover region by increasing the particle density. A more careful analysis, however, shows that ξ_{pair} should actually *decrease with increasing density* starting from the BE limit for small density (“dilute limit”). Consideration of this effect, which originates from the repulsive interaction between the composite bosons, is the main purpose of this paper.

The composite bosons (which form as bound pairs from the constituent fermions in the strong-coupling limit) mutually interact *via* a residual repulsive interaction due to Pauli principle [3,4]. This repulsive interaction, in turn, affects the size of the composite bosons. Specifically, one expects on physical grounds this interaction to *reduce* the size of the composite bosons (with respect to the bound-state radius of a composite boson in isolation), insofar as the repulsive interaction itself decreases for decreasing size of the composite bosons. A new equilibrium size for the composite bosons thus results when combining the effect of the repulsive interaction with the internal energy of a composite boson, the new equilibrium size being smaller than the original size of a composite boson in isolation.

We shall implement this qualitative idea by minimizing the expression of (twice) the fermionic chemical potential $2\mu = -\epsilon + \mu_B$ with respect of the size ξ_{pair} of a composite boson, where ϵ represents the *internal* energy of a composite boson (to be defined below) and μ_B is the bosonic chemical potential determined by the mutual repulsion between the composite bosons [5]. Both ϵ and μ_B are, in fact, functions of ξ_{pair} ; in addition, μ_B depends on the density n .

We shall explicitly verify that the size-shrinking effect is borne out by the analytic zero-temperature mean-field solution for a point-contact interaction both in three [6] and two [7] dimensions. In two dimensions, however, the mean-field density dependence of ξ_{pair} does not agree with what expected in the “dilute limit”. We attribute this difference to the poor treatment of the bosonic scattering

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within the zero-temperature mean field, which rests on the Born approximation as the form of the mean-field bosonic chemical potential implies. Since the Born approximation in two dimensions is known to fail at low energy (which, in turn, corresponds to the “dilute limit” of the Bose gas), the effect of the bosonic interaction on ξ_{pair} cannot be properly treated within the Born approximation.

We will therefore argue that a proper treatment of the residual bosonic interaction (over and above the Born approximation), along the lines recently developed for the three-dimensional case [8], should be especially relevant to the two-dimensional case.

We have focused in this paper on the composite-boson regime of the BCS-BE crossover problem, following previous work on this problem [4,8] which has emphasized the importance of approaching the bosonization *in reverse*, that is, starting from the BE and evolving toward the BCS region. Although this reverse approach is not usually adopted in the literature, we prefer it over the conventional BCS to BE approach which starts from the BCS limit, insofar as the physics of the BE (composite-boson) region is much richer than its counterpart in the BCS limit. Specifically, the reverse approach enables one to perform approximations in the BE limit which would be *a priori* uncontrolled if one would start instead from the BCS limit. These approximations, which are necessarily more involved than those performed in the BCS limit, are nevertheless appropriate also to the BCS limit [8]. In addition, it has been shown that even numerical approximation schemes starting from the BE region are surprisingly accurate also on the BCS side of the crossover, while the reverse is not true [6].

The plan of the paper is as follows. In Section 2 we provide the general physical argument for the size shrinking of the composite bosons with increasing density in the “dilute limit”, irrespective of dimensionality. In Section 3 we confront this general argument with the analytic zero-temperature mean-field results for a contact potential in the strong-coupling limit, both in three and two dimensions. Numerical results for two different finite-range potentials in three dimensions are also shown for comparison. Section 4 gives our conclusions.

2 General argument for the size shrinking of the composite bosons

We consider a system of fermions at zero temperature, mutually interacting *via* an (effective) attractive potential with a finite-range r_0 . For the sake of comparison with the available analytic results in three and two dimensions, we disregard lattice effects and consider the system embedded in a homogeneous background (continuum case).

Composite bosons in isolation are defined when the associated two-body problem admits a bound state (zero-density limit). At small (albeit finite) density, composite bosons are expected to retain their identity in the strong-coupling limit of the original fermionic attraction, *i.e.*, when the binding energy of the associated two-body prob-

lem is much larger than the strength of the *residual interaction* between the composite bosons. Quite generally, this residual interaction has a dominant short-range *repulsive* part due to Pauli principle, which is active among the constituent fermions as soon as the composite bosons overlap. This overlap, in turn, increases (on the average) when the size and/or the density of the composite bosons increase.

More precisely, we shall assume that $k_F a_F \ll 1$, where a_F is the scattering length associated with the fermionic attraction, as well as $a_F \gg r_0$. The former assumption represents a “diluteness” condition, while the second assumption is required to get a well-defined system of composite bosons, for which the Pauli repulsion overwhelms the attractive part of the bosonic potential originating from the finite-range fermionic attraction (see Ref. [8], footnote [25]). In this way, instabilities of the bosonic system (like the one recently pointed out in Ref. [9]) will be suitably avoided.

When the above conditions are satisfied, the composite bosons have a finite size and yet can be considered to be well-defined entities. We assume, therefore, that the standard results for the Bose gas can be used as far as the mutual interaction of the composite bosons is concerned, irrespective of their internal structure. To the internal structure we associate a pair wave function ψ and a corresponding internal energy ϵ , obtained from (minus) the expectation value $\langle \psi | H_2 | \psi \rangle / \langle \psi | \psi \rangle$ of the two-fermion Hamiltonian H_2 over the pair wave function (where H_2 contains the reduced kinetic energy and the attractive two-fermion potential). Physically, $-\epsilon$ represents the energy required to form a composite boson in isolation *with a given* internal wave function ψ for the pair of constituent fermions. We thus regard the energy 2μ required to add *two* fermions to the system as composed of two distinct contributions, namely, (i) the (negative of the) above internal energy $-\epsilon$ and (ii) the energy μ_B required to add eventually the composite boson to the system (with a *finite density* of composite bosons already present). On physical grounds, μ_B is related to the repulsive interaction between the composite bosons and depends on their size and density. Specifically, in three dimensions we take (we set $\hbar = 1$ throughout)

$$\mu_B = \frac{4\pi n_B a_B}{m_B} \quad (1)$$

from the standard theory of the “dilute” Bose gas [10], where $n_B = n/2$ is the bosonic density, $m_B = 2m$ is the mass of a composite boson in terms of the mass m of the constituent fermions, and a_B is the (positive) bosonic scattering length due to the repulsive interaction between the composite bosons. In two dimensions we take instead

$$\mu_B = \frac{4\pi n_B}{m_B \ln \left(\frac{1}{n_B r_0^2} \right)} \quad (2)$$

from the theory of the two-dimensional “dilute” Bose gas [11], where r_0 represents the range of the bosonic interaction. From dimensional considerations, both a_B in

equation (1) and r_o in equation (2) should be proportional to the size ξ_{pair} of the composite bosons, defined by the following functional:

$$\xi_{\text{pair}}^2[\psi] = \frac{\int d\mathbf{r} |\psi(\mathbf{r})|^2 \mathbf{r}^2}{\int d\mathbf{r} |\psi(\mathbf{r})|^2}. \quad (3)$$

We set correspondingly $a_B = \alpha_3 \xi_{\text{pair}}$ and $r_o = \alpha_2 \xi_{\text{pair}}$ in three and two dimensions, in the order, where α_3 and α_2 are positive constants. In this way, the bosonic chemical potential $\mu_B = \mu_B(\xi_{\text{pair}})$ becomes itself a function of ξ_{pair} .

We determine next the function $-\epsilon(\xi_{\text{pair}})$ by minimizing $\langle \psi | H_2 | \psi \rangle / \langle \psi | \psi \rangle$ in the subspace of the wave functions ψ corresponding to a given value of ξ_{pair} . In this way, the equilibrium value $\xi_{\text{pair}}^{(o)}$ and the associated internal energy ϵ_o are obtained from the global minimum of $-\epsilon(\xi_{\text{pair}})$ (ϵ_o coinciding with the binding energy of the associated two-body problem). Near this minimum we set

$$\epsilon(\xi_{\text{pair}}) = \epsilon_o - A_d \left(\xi_{\text{pair}} - \xi_{\text{pair}}^{(o)} \right)^2 \quad (4)$$

where the positive constant A_d depends in general on the dimensionality d . This approximate expression can be used when searching for the minimum of

$$2\mu(\xi_{\text{pair}}) = -\epsilon(\xi_{\text{pair}}) + \mu_B(\xi_{\text{pair}}) \quad (5)$$

to determine the new equilibrium value $\bar{\xi}_{\text{pair}}$ in the presence of a finite (albeit *small*) density of composite bosons. From equation (1) we then obtain for this new equilibrium value in three dimensions:

$$\bar{\xi}_{\text{pair}} = \xi_{\text{pair}}^{(o)} - \frac{2\pi\alpha_3}{m_B A_3} n_B; \quad (6)$$

while from equation (2) we obtain in two dimensions:

$$\bar{\xi}_{\text{pair}} = \xi_{\text{pair}}^{(o)} - \frac{\pi}{m_B A_2 \xi_{\text{pair}}^{(o)} \left[\ln \left(\alpha_2 \xi_{\text{pair}}^{(o)} n_B^{1/2} \right) \right]^2} n_B. \quad (7)$$

Note that the value of $\bar{\xi}_{\text{pair}}$ in the presence of a finite density of composite bosons is *smaller* than the value $\xi_{\text{pair}}^{(o)}$ for a composite boson in isolation, *both* in three and two dimensions. Physically, this shrinking is due to the fact that the decrease of the repulsive bosonic interaction (when the composite bosons contract) prevails over the reduction of the internal energy away from the original equilibrium value ϵ_o .

In three dimensions, by inserting equation (6) into equation (4), the difference $\epsilon(\bar{\xi}_{\text{pair}}) - \epsilon_o$ is seen to decrease quadratically with increasing density. The leading-order correction to the internal energy ϵ is then quadratic in the density. This should be contrasted with the density dependence of the bosonic chemical potential, which is instead linear to the leading order. It can be further checked that adding to equation (1) for the bosonic chemical potential terms of higher order in the small parameter $n_B^{1/3} a_B$, will not modify expression (6) for the leading-order dependence of ξ_{pair} on n_B in the dilute limit. Both the bosonic

chemical potential μ_B and the internal energy ϵ are thus self-consistently calculated by our approach to the leading order in the density.

To leading order, *the binding energy* $-2\mu(\bar{\xi}_{\text{pair}})$ for a composite boson *embedded in the medium* thus *decreases linearly with the density*, owing to the linear dependence of the bosonic chemical potential. This effect has been evidenced in different contexts, namely, for the Bose condensation of excitons in semiconductors [5] and, more recently [12], for the Bose condensation of atomic hydrogen. In the latter case, a linear reduction of the bosonic binding energy has been measured for increasing density, consistently with the general argument presented in this paper.

3 Comparison with 3-d and 2-d mean-field results

In this section, we consider without loss of generality an attractive fermionic point-contact potential, for which the analytic solution of the BCS to BE crossover at the zero-temperature mean-field level is available, both in three [6] and two [7] dimensions. We verify that this solution yields a decrease of ξ_{pair} for increasing density, both in three and two dimensions, in generic agreement with the results of the previous section. Specifically, in three dimensions we are able to recover the mean-field density dependence of ξ_{pair} from the approach of the previous section, by relating a_B to ξ_{pair} *via* the Born approximation. The analogous attempt fails, however, in two dimensions because the Born approximation (which is associated with mean field) strongly overestimates the scattering between composite bosons, thus disrupting the basic assumptions on which the approach of the previous section rests.

We remark that the analytic results presented in this section for a point-contact interaction describe also the general behaviour for a *finite-range* potential in the “dilute” composite-boson regime, which is defined by the two conditions $k_F a_F \ll 1$ and $a_F \gg r_o$ previously mentioned. In this case, the range r_o of the potential is much smaller than the two other length scales in the problem (k_F^{-1} and a_F), so that the mean-field equations for a finite-range potential get always mapped onto the mean-field equations for a point-contact interaction (with the same scattering length). The region where the size-shrinking effect is present (namely, the composite-boson region) coincides thus with the region where the behaviour of the point-contact interaction is “universal”. It is just this coincidence which enables us to establish the size-shrinking effect as a general feature of the strong-coupling limit of the BCS-BE crossover.

3.1 Three-dimensional case

For a three-dimensional point-contact potential, the low-energy fermionic two-body scattering can be conveniently regularized in terms of the scattering length a_F . At finite fermionic density, it is then possible to express all relevant physical quantities (such as the superconducting gap

Δ and the chemical potential) in terms of the dimensionless parameter $k_F a_F$. This can be explicitly verified for the analytic solution of reference [6] given in terms of the complete elliptic integrals. Since $k_F a_F \ll 1$ in the BE limit, physical quantities can further be expanded in powers of $k_F a_F$ in this limit.

In particular, for the pair coherence length one finds from the analytic solution of reference [6]:

$$\xi_{\text{pair}} = \frac{a_F}{\sqrt{2}} \left[1 - \frac{5}{6\pi} (k_F a_F)^3 \right] \quad (8)$$

at the leading order in the bosonic density $n_B = k_F^3/(6\pi^2)$. Equation (8) has been obtained by first expanding equation (30) of reference [6] in powers of $1/x_0$, by then inverting it to obtain x_0 in powers of $k_F a_F$, and by inserting the resulting expression for x_0 into the power expansion of equation (31) of reference [6] for $k_F \xi_{\text{pair}}$. In equation (8) (as well as in Eq. (31) of Ref. [6]) ξ_{pair} is defined according to equation (3), with the BCS choice $\psi_{\text{BCS}}(\mathbf{k}) = \Delta/(2E(\mathbf{k}))$ for the Fourier transform (with wave vector \mathbf{k}) of the pair wave function [1], where $E(\mathbf{k}) = \sqrt{(\mathbf{k}^2/(2m) - \mu)^2 + \Delta^2}$ as usual.

Note from equation (8) that in the zero-density limit ξ_{pair} equals the value $a_F/\sqrt{2}$ of the bound-state radius of a composite boson in isolation. Note also that ξ_{pair} decreases linearly with increasing density of the composite bosons. The mean-field analytic solution for a point-contact interaction thus confirms our general prediction of the size-shrinking effect discussed in the previous section (*cf.* Eq. (6)).

By the same token, for the bosonic chemical potential one finds from the analytic solution of reference [6]:

$$\mu_B = \frac{2a_F k_F^3}{3\pi m} = \frac{8\pi n_B a_F}{m}. \quad (9)$$

Comparison with equation (1) suggests then to identify $a_B = 2a_F$ in the BE (strong-coupling) limit. This result, which was also obtained for the same model system within the fermionic T-matrix approximation in the normal state (*i.e.*, above the superconducting critical temperature) [3,4], identifies $k_F a_F$ with $(3\pi^2/4)^{1/3}$ times the “gas parameter” $n_B^{1/3} a_B$ for a “dilute” Bose gas.

Recall further that the general mapping procedure from the original fermionic system onto the effective bosonic system in the strong-coupling limit (as described in Ref. [4]) provides in three dimensions the value

$$v(0) = \frac{4\pi a_F}{m} \quad (10)$$

for the strength of the “bare” bosonic potential (with all wave vectors and Matsubara frequencies set to zero). One then verifies that the Bogoliubov result

$$\mu_B = n_B v(0) \quad (11)$$

is retrieved by the expression (9) [13].

It is interesting to show that not only the generic linear density dependence but also the coefficients of the expression (8) can be reproduced by the variational principle of the previous section, *provided* we take for α_3 the value $2\sqrt{2}$ (as determined from Eqs. (9, 8) and (1) to leading order in the density) and we assume the BCS form $\psi_{\text{BCS}}(\mathbf{k}) = \Delta/(2E(\mathbf{k}))$ for the Fourier transform of the pair wave function. To this end, it is convenient to express initially $\langle \psi | H_2 | \psi \rangle / \langle \psi | \psi \rangle$ and μ_B as functions of the dimensionless variable $x_o = \mu/\Delta$ instead of ξ_{pair} , to minimize then the resulting expression for 2μ with respect to x_o , and to use eventually the functional relation between x_o and ξ_{pair} (as determined in Ref. [6]) to obtain ξ_{pair} in terms of $k_F a_F$. In the BE limit, where $x_o < 0$ and $|x_o| \gg 1$, this relation reads:

$$k_F \xi_{\text{pair}} = \frac{1}{\sqrt{2}} \left(\frac{3\pi}{16x_o^2} \right)^{1/3} \left(1 - \frac{1}{4x_o^2} \right) \quad (12)$$

to the leading significant orders. One finds:

$$\frac{\langle \psi_{\text{BCS}} | H_2 | \psi_{\text{BCS}} \rangle}{\langle \psi_{\text{BCS}} | \psi_{\text{BCS}} \rangle} = \frac{k_F^2}{2m} \left(\frac{16}{3\pi} \right)^{2/3} \times \left[2|x_o|^{4/3} - 4b|x_o|^{2/3} + \frac{5}{8}|x_o|^{-2/3} - \frac{5}{8}b|x_o|^{-4/3} \right] \quad (13)$$

where we have set $b = (3\pi/16)^{1/3} (k_F a_F)^{-1}$, while $\mu_B = \mu_B(x_o)$ is given by equation (1) with $a_B = 2\sqrt{2}\xi_{\text{pair}}$ and with $\xi_{\text{pair}} = \xi_{\text{pair}}(x_o)$ given by equation (12).

At the leading order, only the first two terms within brackets in equation (13) are relevant to the expression of 2μ , whose minimum is thus located at $|\bar{x}_o| = b^{3/2}$. At the next significant order, all terms within brackets in equation (13) and the leading term in equation (12) are relevant to the expression of 2μ , yielding the new minimum at

$$|\bar{x}_o| = b^{3/2} \left(1 - \frac{9}{64b^3} \right). \quad (14)$$

Inserting equation (14) into equation (12) one recovers eventually the expression (8) for ξ_{pair} , as anticipated. The fact that equation (8), obtained by the mean-field solution, can be reproduced *with the correct numerical coefficients* by the minimization procedure of the previous section (when specialized to the BCS choice for the pair wave function ψ), can be regarded as quite a compelling check on the validity of our general argument of Section 2 and of its underlying assumptions.

Finally, it is interesting to examine the behaviour of ξ_{pair} as a function of the density for a finite-range potential, for which the BCS-BE crossover driven by the density becomes possible [2]. As in reference [2], we consider the separable Nozières-Schmitt-Rink (NSR) potential $V(k, k') = V(1+k^2/k_0^2)^{-1/2}(1+k'^2/k_0^2)^{-1/2}$ (with $k = |\mathbf{k}|$), and the non-separable Gaussian potential $V(\mathbf{k}, \mathbf{k}') = V \exp(-|\mathbf{k} - \mathbf{k}'|^2/k_0^2)$ ($V < 0$ in both cases). For these potentials, an analytic solution for the BCS-BE crossover

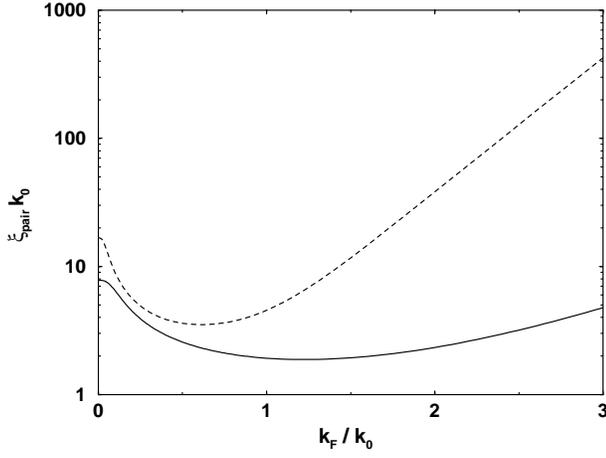


Fig. 1. Pair coherence length (in units of the characteristic length k_0^{-1} of the finite-range fermionic potential) *vs.* k_F/k_0 , for the NSR (full line) and Gaussian (dashed line) potentials. (A typical value of $|V|$ ($= 1.1 V_c$) above the critical value V_c for the existence of a bound state in three dimensions has been considered for both potentials.)

is lacking, even at the mean-field level. We have therefore solved numerically the coupled equations for the gap function $\Delta(k)$ and the chemical potential μ . The pair-coherence length ξ_{pair} has been then determined by using the Fourier transform of equation (3), with the BCS choice for the pair wave function $\psi(k) = \Delta(k)/(2E(k))$. Results are shown in Figure 1, which confirm the presence of the size-shrinking effect in the “universal” composite-boson region. Note, however, that the size-shrinking effect gradually disappears by increasing further the density: ξ_{pair} reaches a minimum at $k_F \sim k_0$ and then starts increasing with the density. The eventual increase of ξ_{pair} with k_F , when $k_F \gg k_0$, is a characteristic feature of a finite-range potential, as it can be readily verified from the asymptotic expressions (4) and (5) of reference [2] for the gap $\Delta(k_F)$ in the limit $k_F \gg k_0$. In this limit, as soon as the system enters the BCS regime, we may use the BCS result $\xi_{\text{pair}} \sim k_F/\Delta(k_F)$ to obtain $\xi_{\text{pair}} \sim k_F^{-1} \exp\left(\frac{k_F A}{|V| m k_0^2}\right)$, where A is a numerical factor different for the NSR and Gaussian potentials. For both potentials, ξ_{pair} is seen to increase exponentially with k_F for large k_F , as far as k_0 is finite. The competition between this asymptotic behaviour of ξ_{pair} when $k_F \gg k_0$ and the initial size-shrinking when $k_F \ll k_0$, leads then to a minimum of ξ_{pair} when $k_F \sim k_0$.

3.2 Two-dimensional case

In two dimensions, the zero-temperature mean-field expression for ξ_{pair} is reported in Appendix B of reference [6] in terms of the available analytic solution [7]. In particular, at the leading significant order in the BE limit

one finds:

$$\xi_{\text{pair}}^2 = \frac{2}{3m\epsilon_0} \left[1 - \frac{38}{15} \left(\frac{4\pi n_B}{m_B \epsilon_0} \right) \right]. \quad (15)$$

For given value of ϵ_0 , ξ_{pair} is thus seen to decrease linearly with the density in the BE limit. One also finds for the bosonic potential:

$$\mu_B = \frac{8\pi n_B}{m_B} \quad (16)$$

which is proportional to the density but independent from the bosonic size. Note from equation (16) that the Bogoliubov result $\mu_B = n_B v(0)$ is retrieved by the mean-field calculation even in two dimensions, since the strength $v(0)$ of the “bare” bosonic potential is given by

$$v(0) = \frac{8\pi}{m_B}, \quad (17)$$

as it can be explicitly verified by applying the prescriptions of reference [4] to the fermion-boson mapping in the two-dimensional case.

The mean-field expressions (15, 16) differ from the “dilute” Bose gas expressions (7, 2), respectively, as they lack the logarithmic term in the denominator. Notwithstanding the decrease of the size of the composite bosons for increasing density obtained by equation (15), the two-dimensional mean-field results appear thus to contradict the picture of Section 2 for a “dilute” gas of composite bosons. In particular, if one would use the Bogoliubov expression (16) to implement the argument of Section 2, the failure of the strength of the “bare” bosonic potential in two dimensions to depend on the size of the composite bosons would not make it energetically convenient to shrink the bosonic size at finite density. The size shrinking obtained by equation (15) is therefore not consistent with the general argument developed in Section 2.

We attribute the difference between the mean-field and the “dilute” gas results to the poor treatment of the boson-boson scattering within the zero-temperature mean field, which rests on the Born approximation as the form (16) of the bosonic chemical potential implies. Let us, in fact, analyze the above results in terms of the outcomes of potential scattering theory in two dimensions [14]. The Born approximation gives for the (dimensionless) low-energy scattering amplitude $f_{\text{Born}}^{(2)} \sim -mv(0)$, where $v(0) \sim v_0 r_0^2$ is typically proportional to the average strength v_0 and range r_0 of the potential. The *exact* low-energy result for the two-dimensional scattering amplitude $f^{(2)} \sim -1/\ln(kr_0)$, on the other hand, is independent of v_0 and vanishes for vanishing wave vector ($k \rightarrow 0$). In two dimensions, therefore, the effect of summing an infinite number of repeated scatterings *drastically modifies* the functional dependence of the scattering amplitude on the strength and range of the potential, and the perturbation theory for the scattering amplitude breaks down [15]. In contrast, in three dimensions $f_{\text{Born}}^{(3)} \sim -mv(0) \sim -a_{\text{Born}}$ where $v(0) \sim v_0 r_0^3$ and a_{Born} is the Born scattering length, whereas $f^{(3)} \sim -a$ is the *exact* low-energy result (a here

being the full scattering length). That is, in three dimensions going from the Born approximation to the exact low-energy result merely changes the numerical value of the scattering length.

For the composite bosons we are specifically interested in, the strength v_0 must be proportional to the binding energy ϵ_0 for dimensional reasons, and the only available length scale is proportional to $(m\epsilon_0)^{-1/2}$. This yields $v(0) \sim \epsilon_0(m\epsilon_0)^{-d/2}$, which corresponds to equations (10) and (17) for $d = 3$ and $d = 2$, respectively, implying that the characteristic strength and range of the potential cannot be independently varied for the composite bosons. It is this feature which, in turn, makes $v(0)$ independent from the size of the composite bosons in two dimensions. The Born approximation to the two-dimensional scattering amplitude for the composite bosons, therefore, not only lacks the functional form of the exact low-energy result, but even fails to yield any dependence on the size of the composite bosons whatsoever.

From the above considerations, it is thus evident that in two dimensions the scattering rate between the composite bosons is overestimated within the Born approximation with respect to the true low-density result, in such a way that the “dilute limit” cannot be achieved at fixed density. Consequently, the physical picture adopted in Section 2 of a system of well-defined weakly interacting bosonic entities is bound to break down within mean field.

4 Concluding remarks

In this paper, we have discussed the size shrinking of composite bosons as a general physical result occurring in the “dilute limit”. We have also verified that this general result is correctly reproduced by the zero-temperature mean-field treatment in three but not in two dimensions. We have related the failure in two dimensions to the break down of the Born approximation at low energy. In addition, we have noted a peculiar relation between the effective strength and range of the potential acting between composite bosons.

For the three-dimensional case, the relevance of including the mutual interaction between the composite bosons in the strong-coupling (BE) limit has been emphasized some time ago in references [5,16]; more recently, the significance of treating the scattering between the composite bosons beyond the Born approximation has been addressed in reference [8]. We have argued here, however, that the failure to account for the effects of the bosonic interaction is more severe in two than in three dimensions, because the Born approximation breaks completely down in two dimensions. By the same token, we also expect that the boundary in the phase diagram between the Bose-Einstein and crossover regions, discussed in reference [2] within mean field, should be significantly modified by a proper treatment of the bosonic interaction, along the lines recently developed for the three-dimensional case [8].

It is further clear that the criticisms recently raised to the fermionic T-matrix approximation when applied to the normal phase in three dimensions (which in the strong-coupling limit has been proved to reduce to the Born approximation as far as the scattering between composite bosons is concerned [8]) are even more appropriate for the two-dimensional case where the Born approximation fails completely. This remark makes it somewhat questionable the use of the fermionic (self-consistent) T-matrix approximation in two dimensions [17,18], to describe the tendency toward the formation of preformed pairs above the superconducting critical temperature, at least as the strong-coupling (bosonic) limit is approached.

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