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Evolution from BCS superconductivity to Bose condensation: analytic results for the crossover in three dimensions

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Abstract. We provide an analytic solution for the mean-field equations and for the relevant physical quantities at the Gaussian level, in terms of the complete elliptic integrals of the first and second kinds, for the crossover problem from BCS superconductivity to Bose-Einstein condensation of a *three*-dimensional system of free fermions interacting *via* an attractive contact potential at zero temperature. This analytic solution enables us to follow the evolution between the two limits in a particularly simple and transparent way, as well as to verify the absence of singularities during the evolution.

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1 Introduction

The interest in the crossover from BCS superconductivity to Bose-Einstein (BE) condensation has quite recently increased both experimentally and theoretically after the appearance of new angular resolved photoemission data on underdoped cuprate samples, which indicate the persistence of a gap in the single-particle excitation spectrum at temperatures well above the superconducting critical temperature [1–3]. Previous to these data the suggestion that superconductivity in cuprates (as well as in other "exotic" materials) might require an intermediate approach between the two (BCS and BE) limits has been emphasized especially by Uemura [4].

From the theoretical point of view, evolution from BCS superconductivity to BE condensation has been considered originally by Nozières and Schmitt-Rink [5], following the pioneering works by Eagles [6] and Leggett [7]. More recently, the crossover problem was addressed in references [8–14], prompted by the experimental suggestion that the superconducting coherence length in cuprates is considerably shorter than in conventional superconductors. In all these works known results were recovered analytically in the two (BCS and BE) limits. However, no analytic result was obtained for the most interesting crossover region, which had thus to be treated numerically (with the sole exception of the two-dimensional case considered in Ref. [15]).

In this paper we provide an analytic solution in the *three dimensional* case covering the *whole* crossover region for a number of physical quantities evaluated at the mean-field level and with the inclusion of Gaussian fluctuations, by considering a system of fermions in free space at zero temperature mutually interacting *via* an attractive *contact* potential. For this case we are, in fact, able to express these physical quantities in terms of the complete elliptic integrals of the first and second kinds, whose analytic properties and numerical values are extensively tabulated [16]. Our solution enables us to interpolate in a rigorous fashion between the two (BCS and BE) limits, thus avoiding the problems which occur with a full numerical approach. Although our analytical solution has been unavoidably obtained with simplifying assumptions (namely, at zero temperature and using a free-particle dispersion relation and a contact interaction potential), it might nevertheless be regarded as a reference solution with which numerical solutions obtained for more complicated cases could be compared.

A few words of comments about the validity of the continuum model we are using and of the Gaussian approximation we are adopting are in order at this point. It is known [5,12] that the continuum model gives a sensible description even of the BE limit and that this model can be recovered from the Hubbard model away from half filling of the band (*i.e.*, in the low-density regime). In addition, it is known that the Gaussian approximation is valid in the BCS limit, where fluctuations about the mean field are small, as well as in the BE limit, where it corresponds to the Bogoliubov treatment of the interacting Bose gas (*i.e.*, the low-density limit) [5,10–12]. In the intermediate (crossover) regime, on the other hand, where there is admittedly no small parameter to control the approximations, the Gaussian approximation should be thought to

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provide an interpolation scheme to connect the two (BCS and BE) limits.

The plan of the paper is as follows. In Section 2 we express a number of mean-field quantities (such as the chemical potential μ, the gap parameter Δ, the pair-correlation length $\xi_{\rm pair}$, and the bound-state energy ϵ_0) in terms of the complete elliptic integrals of the first and second kinds with argument $\kappa^2 = (\sqrt{1+x_0^2}+x_0)/(2\sqrt{1+x_0^2})$ (where $x_0 = \mu/\Delta$) ranging from 0 (BE limit) to 1 (BCS limit). In Section 3 our analysis is extended to quantities such as the phase coherence length $\xi_{\rm phase}$ and the sound velocity s, by considering the Gaussian fluctuations about the mean field. To keep the presentation compact, the properties of the elliptic integrals used in our treatment are summarized in Appendix A. In Appendix B we report for the sake of comparison the solution for the two-dimensional case given previously by reference [15].

2 Analytic results at the mean-field level

We consider the Hamiltonian ($\hbar = 1$)

$$H = \sum_\sigma \int \mathrm{d}\mathbf{r}\, \psi_\sigma^\dagger(\mathbf{r}) \left(-\frac{\nabla^2}{2m} - \mu\right) \psi_\sigma(\mathbf{r}) + g \int \mathrm{d}\mathbf{r}\, \psi_\uparrow^\dagger(\mathbf{r})\psi_\downarrow^\dagger(\mathbf{r})\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r}) \quad (1)$$

where $\psi_\sigma(\mathbf{r})$ is the fermionic field operator with spin projection σ, m is the fermionic (effective) mass, and $g = V\Omega$ is the strength of the short-range (contact) potential between fermions with $V < 0$ (Ω being the volume occupied by the system).

At the zero temperature, the mean-field equations for the gap parameter Δ and the chemical potential μ are obtained by a suitable decoupling of the Hamiltonian (1) and are given by [17]

$$-\frac{1}{V} = \sum_{\mathbf{k}} \frac{1}{2E_{\mathbf{k}}} \quad (2)$$

$$n = \frac{N}{\Omega} = \frac{2}{\Omega}\sum_{\mathbf{k}} v_{\mathbf{k}}^2 \quad (3)$$

where $\mathbf{k}$ is the wave vector, N the total number of fermions, and

$$\xi_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m} - \mu,\ E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2},\ v_{\mathbf{k}}^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right). \quad (4)$$

Owing to our choice of a contact potential, equation (2) diverges in the ultraviolet (both in two and three dimensions) and requires a suitable regularization. In three dimensions it is common practice to introduce the scattering amplitude $a_{\rm s}$ defined *via* the equation [10–12]

$$\frac{m}{4\pi a_{\rm s}} = \frac{1}{\Omega V} + \frac{1}{\Omega}\sum_{\mathbf{k}} \frac{m}{\mathbf{k}^2}, \quad (5)$$

where the divergent sum on the right-hand side of equation (5) results in a finite value of $a_{\rm s}$ by letting $V \to 0$ in a suitable way. Subtracting equation (5) from equation (2) we obtain

$$-\frac{m}{4\pi a_{\rm s}} = \frac{1}{\Omega}\sum_{\mathbf{k}}\left(\frac{1}{2E_{\mathbf{k}}} - \frac{m}{\mathbf{k}^2}\right) \quad (6)$$

which has to be solved simultaneously with equation (3) to determine Δ and μ as functions of $a_{\rm s}$.

In three dimensions it is convenient to introduce the following dimensionless quantities

$$\begin{cases} x^2 = \dfrac{k^2}{2m}\dfrac{1}{\Delta}\ , & x_0 = \dfrac{\mu}{\Delta}\ , \\ \xi_x = \dfrac{\xi_k}{\Delta} = x^2 - x_0 & , E_x = \dfrac{E_k}{\Delta} = \sqrt{\xi_x^2 + 1} \end{cases} \quad (7)$$

with $k = |\mathbf{k}|$, and express equations (3, 6) in the form

$$-\frac{1}{a_{\rm s}} = \frac{2}{\pi}(2m\Delta)^{1/2} I_1(x_0) \quad (8)$$

$$n = \frac{1}{2\pi^2}(2m\Delta)^{3/2} I_2(x_0) \quad (9)$$

where

$$I_1(x_0) = \int_0^\infty \mathrm{d}x\, x^2 \left(\frac{1}{E_x} - \frac{1}{x^2}\right) \quad (10)$$

$$I_2(x_0) = \int_0^\infty \mathrm{d}x\, x^2 \left(1 - \frac{\xi_x}{E_x}\right). \quad (11)$$

The integrals (10, 11) were originally considered by Eagles [6], who evaluated them numerically as functions of the crossover parameter x_0 ranging from $-\infty$ (BE limit) to $+\infty$ (BCS limit). We shall soon show that the integrals (10, 11) can actually be evaluated in a closed form for all values of x_0. Note that $I_2(x_0) \geq 0$ while $I_1(x_0)$ can take both signs. It is relevant to mention at this point that equation (6) was already integrated analytically in reference [11] in terms of hypergeometric functions; we have verified that the solution for that equation given in reference [11] can also be expressed in terms of the complete elliptic integrals, in agreement with equation (30) below.

Before proceeding further, it is convenient to render equations (2) dimensionless by introducing the Fermi energy $\epsilon_{\rm F} = k_{\rm F}^2/2m = (3\pi^2 n)^{2/3}/2m$ where $k_{\rm F}$ is the Fermi momentum. In this way equation (9) becomes

$$\frac{\Delta}{\epsilon_{\rm F}} = \left[\frac{2}{3I_2(x_0)}\right]^{2/3} \quad (12)$$

and equation (8) reduces to

$$\frac{1}{k_{\rm F} a_{\rm s}} = -\frac{2}{\pi}\left[\frac{2}{3I_2(x_0)}\right]^{1/3} I_1(x_0). \quad (13)$$

Note that the right-hand sides of equations (12, 13) depend on x_0 only. Equation (13) can thus be inverted to obtain x_0 as a function of $k_F a_s$; from equation (12) and from $\mu/\epsilon_F = x_0 \Delta/\epsilon_F$ one can then obtain the two parameters Δ/ϵ_F and μ/ϵ_F as functions of $k_F a_s$.

Alternatively, one can use $k_F\xi_{pair}$ as the independent variable in the place of $k_F a_s$, where ξ_{pair} is the characteristic length for pair correlation given by the following expression in three dimensions at the mean-field level [12]:

$$\xi_{pair}^2 = \frac{1}{m^2}\frac{\int_0^\infty dk (k^4\xi_k^2/E_k^6)}{\int_0^\infty dk (k^2/E_k^2)} = \frac{2}{m\Delta}\frac{I_3(x_0)}{I_4(x_0)} \quad (14)$$

where

$$I_3(x_0) = \int_0^\infty dx \frac{x^4\xi_x^2}{E_x^6} \quad (15)$$

$$I_4(x_0) = \int_0^\infty dx \frac{x^2}{E_x^2} \quad (16)$$

are two additional integrals expressed in terms of the quantities (7). Contrary to the integrals I_1 and I_2, the integrals I_3 and I_4 are elementary and can be evaluated *via* the residues technique. One obtains:

$$I_3(x_0) = \frac{\pi}{16}\frac{x_1(1+x_1^4)}{(1+x_0^2)^{1/2}} \quad (17)$$

$$I_4(x_0) = \frac{\pi}{2}x_1 \quad (18)$$

with the notation

$$x_1^2 = \frac{\sqrt{1+x_0^2}+x_0}{2}. \quad (19)$$

Making use of equation (12) we obtain eventually

$$(k_F\xi_{pair})^2 = \frac{\epsilon_F}{2\Delta}\frac{(1+x_1^4)}{(1+x_0^2)^{1/2}} = \frac{(1+x_1^4)}{2(1+x_0^2)^{1/2}}\left[\frac{3I_2(x_0)}{2}\right]^{2/3}. \quad (20)$$

In this way equation (13) can be dropped in favor of equation (20), which can be inverted to obtain x_0 as a function of $k_F\xi_{pair}$.

There remains to evaluate the integrals $I_1(x_0)$ entering equation (13) and $I_2(x_0)$ entering equations (12, 20). To this end, we introduce the auxiliary integrals

$$I_5(x_0) = \int_0^\infty dx \frac{x^2}{E_x^3}, \quad (21)$$

$$I_6(x_0) = \int_0^\infty dx \frac{x^2\xi_x}{E_x^3}, \quad (22)$$

such that

$$I_1(x_0) = 2\,(x_0 I_6(x_0) - I_5(x_0)) \quad (23)$$

$$I_2(x_0) = \frac{2}{3}\,(x_0 I_5(x_0) + I_6(x_0)) \quad (24)$$

after integration by parts. The auxiliary integrals $I_5(x_0)$ and $I_6(x_0)$ can, in turn, be expressed as linear combinations of the complete elliptic integrals of the first $[F(\frac{\pi}{2},\kappa)]$ and second $[E(\frac{\pi}{2},\kappa)]$ kinds. Reduction of a generic integral of the elliptic kind to the normal Legendre's form has been treated at length in the literature [18]. Here we proceed as follows. Integration by parts gives for $I_6(x_0)$ (*cf.* Eq. (22)):

$$\begin{aligned} I_6(x_0) &= -\frac{1}{2}\int_0^\infty dx\, x \frac{d}{dx}\frac{1}{E_x} = \frac{1}{2}\int_0^\infty dx \frac{1}{E_x} \\ &= \frac{1}{2}\int_0^\infty dx \frac{1}{(x^4 - 2x_0x^2 + x_0^2 + 1)^{1/2}} \\ &= \frac{1}{2(1+x_0^2)^{1/4}}F(\frac{\pi}{2},\kappa) \end{aligned} \quad (25)$$

where in the last line use has been made of the results of Appendix A and where ($0 \le \kappa^2 < 1$)

$$\kappa^2 = \frac{x_1^2}{(1+x_0^2)^{1/2}} \quad (26)$$

with x_1 given by equation (19). For $I_5(x_0)$ we obtain instead (*cf.* Eq. (21)):

$$\begin{aligned} I_5(x_0) &= \int_0^\infty dx \frac{x^2}{(x^4 - 2x_0x^2 + x_0^2 + 1)^{3/2}} \\ &= (1+x_0^2)^{1/4}E(\frac{\pi}{2},\kappa) \\ &\quad - \frac{1}{4x_1^2(1+x_0^2)^{1/4}}F(\frac{\pi}{2},\kappa) \end{aligned} \quad (27)$$

where again use has been made of the results of Appendix A.

In conclusion, we obtain for the quantities of interest (*cf.* Eqs. (12, 13, 20)):

$$\frac{\Delta}{\epsilon_F} = \frac{1}{(x_0I_5(x_0) + I_6(x_0))^{2/3}} \quad (28)$$

$$\frac{\mu}{\epsilon_F} = \frac{\mu}{\Delta}\frac{\Delta}{\epsilon_F} = \frac{x_0}{(x_0I_5(x_0) + I_6(x_0))^{2/3}} \quad (29)$$

$$\frac{1}{k_F a_s} = -\frac{4}{\pi}\frac{(x_0I_6(x_0) - I_5(x_0))}{(x_0I_5(x_0) + I_6(x_0))^{1/3}} \quad (30)$$

$$k_F\xi_{pair} = \left(\frac{1+x_1^4}{2}\right)^{1/2}\frac{(x_0I_5(x_0) + I_6(x_0))^{1/3}}{(1+x_0^2)^{1/4}} \quad (31)$$

with $I_5(x_0)$ and $I_6(x_0)$ given by equations (27, 25), respectively. It is sometimes convenient to normalize the chemical potential, when negative, with respect to the bound-state energy ϵ_0 of the associated two-fermion problem. In the three-dimensional case, ϵ_0 can be expressed in terms of a_s whenever $a_s \ge 0$. One finds [12]

$$\epsilon_0 = \frac{1}{ma_s^2}, \quad (32)$$

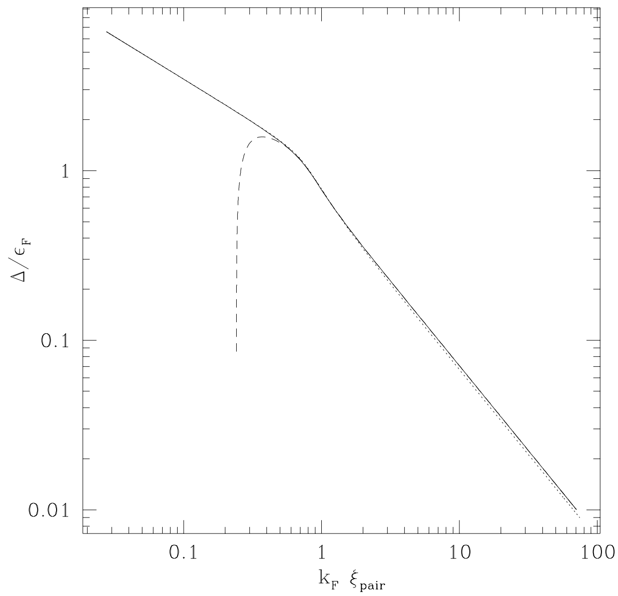

Fig. 1. $\Delta/\epsilon_{\rm F}$ *versus* $k_{\rm F}\xi_{\rm pair}$, obtained from equations (28, 31). Full curve: exact solution; dashed curve: BCS approximation, obtained by including only the first two terms in equations (81, 82); dotted curve: BE approximation, obtained by including only the first two terms in equations (79, 80).

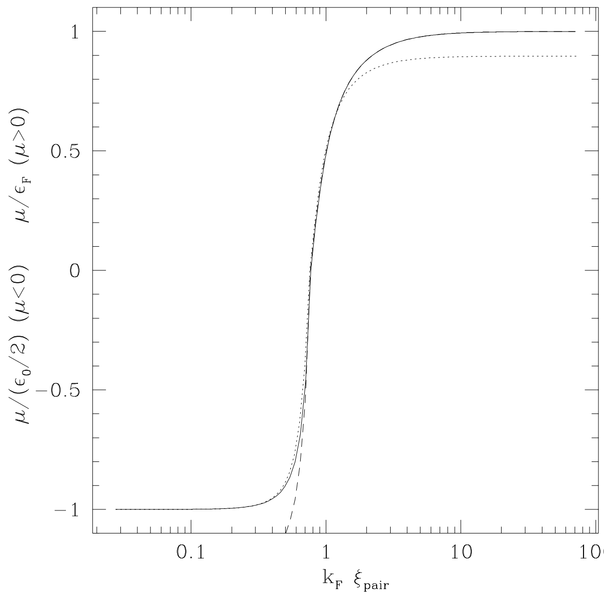

Fig. 2. $\mu/\epsilon_{\rm F}$ for $\mu > 0$ and $\mu/(\epsilon_0/2)$ for $\mu < 0$ *versus* $k_{\rm F}\xi_{\rm pair}$, obtained from equations (29, 31, 33). Conventions are as in Figure 1.

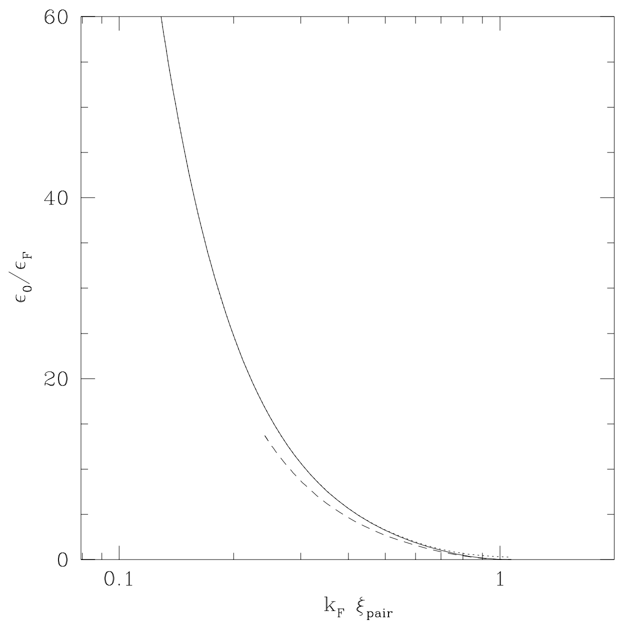

Fig. 3. $\epsilon_0/\epsilon_{\rm F}$ *versus* $k_{\rm F}\xi_{\rm pair}$, obtained from equations (31, 33) when $a_{\rm s}$ given by equation (30) is positive. Conventions are as in Figure 1.

so that (*cf.* Eq. (30))

$$\frac{\epsilon_0}{\epsilon_{\rm F}} = \frac{2}{(k_{\rm F}a_{\rm s})^2} = \frac{32}{\pi^2}\frac{(x_0 I_6(x_0) - I_5(x_0))^2}{(x_0 I_5(x_0) + I_6(x_0))^{2/3}}. \qquad (33)$$

Numerical values of the complete elliptic integrals F and E have been extensively tabulated [16]. Otherwise, one may generate them with the required accuracy *via* equations (79-82). In this way, the desired values of $I_5(x_0)$ and $I_6(x_0)$ can be obtained for given x_0 (or for given $k_{\rm F}\xi_{\rm pair}$ by inverting Eq. (31)). In Figures 1-3 we report, respectively, the values of $\Delta/\epsilon_{\rm F}$, $\mu/\epsilon_{\rm F}$ for $\mu > 0$ and $\mu/(\epsilon_0/2)$ for $\mu < 0$, and $\epsilon_0/\epsilon_{\rm F}$ *versus* $k_{\rm F}\xi_{\rm pair}$ obtained by this procedure. Within numerical accuracy, all values coincide with those calculated by solving numerically the gap equation and the normalization condition (3) for the limiting case of a contact potential [12].

Besides the exact result (full curve), Figures 1-3 show for comparison two additional curves obtained by approximating, respectively, the elliptic integrals F and E by the first two terms of equations (79, 80) (dotted curve) and by the first two terms of equations (81, 82) (dashed curve). In principle, these approximate results are expected to be reliable in the BE and BCS limits, in the order. Note, however, that the BE approximate result is surprisingly accurate on the BCS side of the crossover.

In the BCS and BE limits the values of $I_5(x_0)$ and $I_6(x_0)$ can be obtained by retaining only a few significant terms in the expansions (79-82). In particular, in the BCS limit $x_0 \gg 1$ so that $\kappa^2 \simeq 1 - 1/(4x_0^2)$ and

$$\begin{cases} I_5(x_0) & \simeq \sqrt{x_0} \\ I_6(x_0) & \simeq \ln x_0/(2\sqrt{x_0}). \end{cases} \qquad (34)$$

Then:

$$\begin{cases} \Delta/\epsilon_{\rm F} & \simeq 1/x_0 \\ \mu/\epsilon_{\rm F} & \simeq 1 \\ 1/(k_{\rm F}a_{\rm s}) & \simeq -(2/\pi)\ln x_0 \\ k_{\rm F}\xi_{\rm pair} & \simeq x_0/\sqrt{2}. \end{cases} \qquad (35)$$

In the BE limit, on the other hand, $x_0 < 0$ and $|x_0| \gg 1$ so that $\kappa^2 = 1/(4x_0^2)$ and

$$\begin{cases} I_5(x_0) & \simeq \pi/(16|x_0|^{3/2}) \\ I_6(x_0) & \simeq \pi/(4|x_0|^{1/2}). \end{cases} \tag{36}$$

Then:

$$\begin{cases} \Delta/\epsilon_{\mathrm{F}} & \simeq [16/(3\pi)]^{2/3}|x_0|^{1/3} \\ \mu/\epsilon_{\mathrm{F}} & \simeq -[16/(3\pi)]^{2/3}|x_0|^{4/3} \\ 1/(k_{\mathrm{F}}a_{\mathrm{s}}) & \simeq [16/(3\pi)]^{1/3}|x_0|^{2/3} \\ \epsilon_0/\epsilon_{\mathrm{F}} & \simeq 2[16/(3\pi)]^{2/3}|x_0|^{4/3} \\ k_{\mathrm{F}}\xi_{\mathrm{pair}} & \simeq \frac{1}{\sqrt{2}}[16/(3\pi)]^{-1/3}|x_0|^{-2/3}. \end{cases} \tag{37}$$

The limiting BCS (35) and BE (37) values coincide with those calculated previously by different methods [12].

We mention, finally, that another quantity which can be evaluated analytically at the mean-field level for all values of x_0 is the single-particle density of states.

All the above results hold for the three-dimensional system. Analogous results for the two-dimensional system are straightforwardly expressed in terms of elementary integrals and are reported for comparison in Appendix B.

3 Analytic results at the Gaussian level

Besides the quantities of Section 2 defined at the mean-field level, additional quantities whose definition requires the introduction of Gaussian fluctuations can also be expressed analytically in three dimensions at zero temperature for the Hamiltonian (1), in terms of the complete elliptic integrals F and E for *all* values of the parameter x_0 (*i.e.*, following the evolution from BCS to BE). In particular, we shall consider the phase coherence length ξ_{phase} (associated with the spatial fluctuations of the superconducting order parameter) and the sound velocity s (associated with the Goldstone mode of the broken symmetry).

The matrix of the Gaussian fluctuations has elements [12,19]

$$\Gamma(\mathbf{q},\omega) = \begin{pmatrix} A(\mathbf{q},\omega) & B(\mathbf{q},\omega) \\ B(\mathbf{q},\omega) & A(-\mathbf{q},-\omega) \end{pmatrix} \tag{38}$$

where ω is the frequency and (for a real order parameter)

$$A(\mathbf{q},\omega) = \frac{1}{\Omega}\sum_{\mathbf{k}}\left(\frac{1}{2E_{\mathbf{k}}} - \frac{u_{\mathbf{k}}^2 u_{\mathbf{k-q}}^2}{E_{\mathbf{k}} + E_{\mathbf{k-q}} - \omega - i\eta} - \frac{v_{\mathbf{k}}^2 v_{\mathbf{k-q}}^2}{E_{\mathbf{k}} + E_{\mathbf{k-q}} + \omega + i\eta}\right) \tag{39}$$

$$B(\mathbf{q},\omega) = \frac{1}{\Omega}\sum_{\mathbf{k}} u_{\mathbf{k}}v_{\mathbf{k}}u_{\mathbf{k-q}}v_{\mathbf{k-q}}\left(\frac{1}{E_{\mathbf{k}} + E_{\mathbf{k-q}} - \omega - i\eta} + \frac{1}{E_{\mathbf{k}} + E_{\mathbf{k-q}} + \omega + i\eta}\right) \tag{40}$$

at zero temperature. In these expressions, $u_{\mathbf{k}}^2 = 1 - v_{\mathbf{k}}^2$ and η is a positive infinitesimal. To extract ξ_{phase} and s we need consider the expansion of $A(\mathbf{q},\omega)$ and $B(\mathbf{q},\omega)$ for small values of $|\mathbf{q}|$ and ω, such that

$$\frac{\mathbf{q}^2}{2m} \ll \omega^* \qquad \text{and} \qquad \omega \ll \omega^* \tag{41}$$

where

$$\omega^* = \begin{cases} 2\Delta, & \mu > 0 \\ 2\sqrt{\Delta^2 + \mu^2}, & \mu < 0. \end{cases} \tag{42}$$

The result is:

$$A(\mathbf{q},\omega) = a_0 + a_1\,\omega + a_2\,\mathbf{q}^2 + a_3\,\omega^2 + \dots \tag{43}$$
$$B(\mathbf{q},\omega) = b_0 + b_2\,\mathbf{q}^2 + b_3\,\omega^2 + \dots, \tag{44}$$

with

$$a_0 = \frac{1}{\Omega}\sum_{\mathbf{k}}\frac{\Delta^2}{4E_{\mathbf{k}}^3} \tag{45}$$

$$a_1 = -\frac{1}{\Omega}\sum_{\mathbf{k}}\frac{\xi_{\mathbf{k}}}{4E_{\mathbf{k}}^3} \tag{46}$$

$$a_2 = \frac{1}{\Omega}\sum_{\mathbf{k}}\frac{1}{32m}\left\{\frac{\xi_{\mathbf{k}}(2\xi_{\mathbf{k}}^2 - \Delta^2)}{E_{\mathbf{k}}^5} + \frac{\mathbf{k}^2}{dm}\frac{\Delta^2(8\xi_{\mathbf{k}}^2 + 3\Delta^2)}{E_{\mathbf{k}}^7}\right\} \tag{47}$$

$$a_3 = -\frac{1}{\Omega}\sum_{\mathbf{k}}\frac{2\xi_{\mathbf{k}}^2 + \Delta^2}{16E_{\mathbf{k}}^5} \tag{48}$$

and

$$b_0 = a_0 \tag{49}$$

$$b_2 = \frac{1}{\Omega}\sum_{\mathbf{k}}\frac{1}{32m}\left\{-\frac{3\xi_{\mathbf{k}}\Delta^2}{E_{\mathbf{k}}^5} + \frac{\mathbf{k}^2}{dm}\frac{\Delta^2(2\xi_{\mathbf{k}}^2 - 3\Delta^2)}{E_{\mathbf{k}}^7}\right\} \tag{50}$$

$$b_3 = \frac{1}{\Omega}\sum_{\mathbf{k}}\frac{\Delta^2}{16E_{\mathbf{k}}^5} \tag{51}$$

(d being the dimensionality). In particular, to determine ξ_{phase} one has to consider the quantity [12]

$$A(\mathbf{q},\omega=0) + B(\mathbf{q},\omega=0) = 2a_0 + (a_2 + b_2)\mathbf{q}^2 + \dots \tag{52}$$

from which

$$\xi_{\mathrm{phase}}^2 = \frac{a_2 + b_2}{2a_0}\,. \tag{53}$$

To determine the sound velocity s one has instead to consider the full determinant

$$A(\mathbf{q},\omega)A(\mathbf{q},-\omega) - B(\mathbf{q},\omega)^2 = = 2a_0(a_2 - b_2)\mathbf{q}^2 + [2a_0(a_3 - b_3) - a_1^2]\omega^2 + \dots \tag{54}$$

which vanishes for $\omega = \omega(\mathbf{q}) = s|\mathbf{q}|$, with

$$s^2 = \frac{2a_0(a_2 - b_2)}{2a_0(b_3 - a_3) + a_1^2} \cdot \tag{55}$$

We are left with evaluating the integrals entering equations (53, 55). In *three* dimensions we obtain:

$$a_0 = \frac{m}{(2\pi)^2}(2m\Delta)^{1/2} I_5(x_0), \tag{56}$$

$$a_1 = -\frac{m}{(2\pi)^2}\left(\frac{2m}{\Delta}\right)^{1/2} I_6(x_0), \tag{57}$$

$$a_2 + b_2 = \frac{1}{(2\pi)^2}\frac{1}{4}\left(\frac{2m}{\Delta}\right)^{1/2}\left[\int_0^\infty \mathrm{d}x \frac{x^2\xi_x^3}{E_x^5} - 2\int_0^\infty \mathrm{d}x \frac{x^2\xi_x}{E_x^5} + \frac{10}{3}\int_0^\infty \mathrm{d}x \frac{x^4\xi_x^2}{E_x^7}\right], \tag{58}$$

$$a_2 - b_2 = \frac{1}{(2\pi)^2}\frac{1}{4}\left(\frac{2m}{\Delta}\right)^{1/2}\left[I_6(x_0) + 2\int_0^\infty \mathrm{d}x \frac{x^4}{E_x^5}\right], \tag{59}$$

$$b_3 - a_3 = \frac{1}{(2\pi)^2}\frac{1}{4}\left(\frac{2m}{\Delta}\right)^{3/2} I_5(x_0), \tag{60}$$

where use has been made of the notation (7) and of the integrals (21, 22). The four new integrals appearing in equations (58, 59) can also be expressed as linear combinations of $I_5(x_0)$ and $I_6(x_0)$. Integrations by parts and simple manipulations lead to:

$$\int_0^\infty \mathrm{d}x \frac{x^2\xi_x}{E_x^5} = \frac{1}{6}\int_0^\infty \frac{1}{E_x^3} = \frac{x_0 I_5(x_0) + I_6(x_0)}{6(1 + x_0^2)}; \tag{61}$$

$$\int_0^\infty \mathrm{d}x \frac{x^2\xi_x^3}{E_x^5} = I_6(x_0) - \int_0^\infty \mathrm{d}x \frac{x^2\xi_x}{E_x^5}; \tag{62}$$

$$\int_0^\infty \mathrm{d}x \frac{x^4}{E_x^5} = \int_0^\infty \mathrm{d}x \frac{x^2\xi_x}{E_x^5} + x_0\int_0^\infty \mathrm{d}x \frac{x^2}{E_x^5} = (1 + x_0^2)\int_0^\infty \mathrm{d}x \frac{x^2\xi_x}{E_x^5} + \frac{x_0}{2} I_5(x_0); \tag{63}$$

$$\int_0^\infty \mathrm{d}x \frac{x^4\xi_x^2}{E_x^7} = \frac{3}{10}\int_0^\infty \mathrm{d}x \frac{x^2\xi_x}{E_x^5} + \frac{1}{5}\int_0^\infty \mathrm{d}x \frac{x^4}{E_x^5} \cdot \tag{64}$$

In conclusion we obtain:

$$a_2 + b_2 = \frac{1}{(2\pi)^2}\frac{1}{12}\left(\frac{2m}{\Delta}\right)^{1/2} \times \left\{2I_6(x_0) + \frac{(1 + 4x_0^2)}{3(1 + x_0^2)}[I_6(x_0) + x_0 I_5(x_0)]\right\} \tag{65}$$

and

$$a_2 - b_2 = \frac{1}{(2\pi)^2}\frac{1}{3}\left(\frac{2m}{\Delta}\right)^{1/2}\{I_6(x_0) + x_0 I_5(x_0)\}. \tag{66}$$

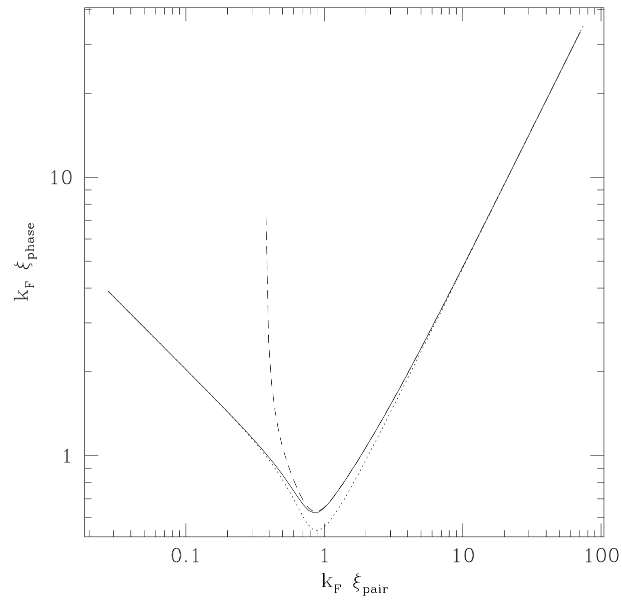

Fig. 4. $k_F\xi_{phase}$ *versus* $k_F\xi_{pair}$, obtained from equations (31, 67). Conventions are as in Figure 1.

Equation (53) thus reduces to

$$(k_F\xi_{phase})^2 = \frac{\epsilon_F}{\Delta}\frac{1}{12I_5(x_0)} \times \left\{2I_6(x_0) + \frac{(1 + 4x_0^2)}{3(1 + x_0^2)}[I_6(x_0) + x_0 I_5(x_0)]\right\} \tag{67}$$

with ϵ_F/Δ given by equation (28), while equation (55) becomes

$$\left(\frac{s}{v_F}\right)^2 = \frac{1}{3}\frac{\Delta}{\epsilon_F}\frac{I_5(x_0)(I_6(x_0) + x_0 I_5(x_0))}{I_5(x_0)^2 + I_6(x_0)^2} \tag{68}$$

where $v_F = k_F/m$.

The expressions (67, 68) provide the desired *analytic* expressions of $k_F\xi_{phase}$ and s/v_F for *all* values of x_0. Using again equations (25, 27) and the tabulated values of the elliptic integrals, we report in Figures 4 and 5 the values of $k_F\xi_{phase}$ and s/v_F, respectively, *versus* $k_F\xi_{pair}$. To within numerical accuracy, we reproduce in this way the numerical results of reference [12] for $k_F\xi_{phase}$. Note, again, that the BE approximation (dotted curve) is surprisingly accurate even on the BCS side.

In the BCS and BE limits one can use the approximate values (34, 36), in the order, for the integrals $I_5(x_0)$ and $I_6(x_0)$. This gives:

$$\begin{cases} k_F\xi_{phase} \simeq x_0/3 \\ s/v_F \simeq 1/\sqrt{3} \end{cases} \tag{69}$$

in the BCS limit, and

$$\begin{cases} k_F\xi_{phase} \simeq (3\pi|x_0|/16)^{1/3} \\ s/v_F \simeq (12\pi|x_0|)^{-1/3} \end{cases} \tag{70}$$

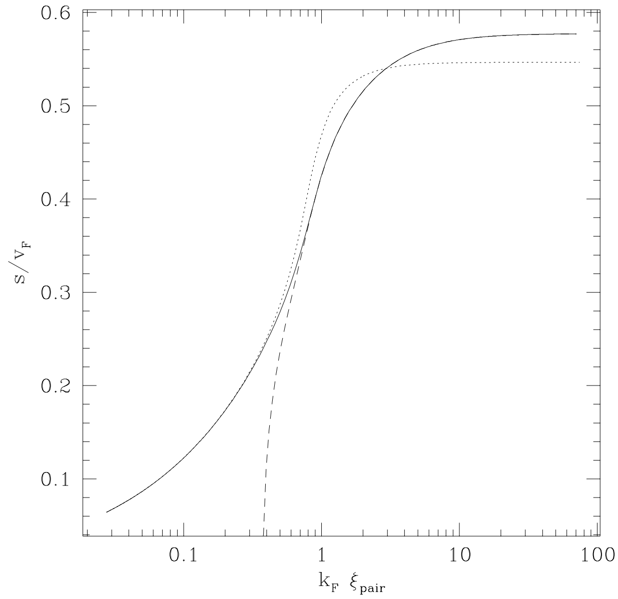

Fig. 5. s/v_{F} *versus* $k_{\mathrm{F}}\xi_{\mathrm{pair}}$, obtained from equations (31, 68). Conventions are as in Figure 1.

in the BE limit. Note that in the BE limit the product

$$s\,\xi_{\mathrm{phase}} = \frac{1}{m}\left(\frac{s}{v_{\mathrm{F}}}\right)(k_{\mathrm{F}}\xi_{\mathrm{phase}}) \simeq \frac{1}{4m} \tag{71}$$

is a constant and coincides with the Bogoliubov result $(2m_{\mathrm{B}})^{-1}$ for composite bosons with mass $m_{\mathrm{B}} = 2m$, thus confirming the general results established in reference [12] for the mapping onto a bosonic system in the strong-coupling limit.

An additional quantity which can be evaluated analytically at the Gaussian level is the coefficient γ of the quartic term in the dispersion relation

$$\omega(\mathbf{q})^2 = s^2\mathbf{q}^2 + \gamma\left(\frac{\mathbf{q}^2}{4m}\right)^2, \tag{72}$$

obtained by expanding the determinant (54) to higher order. Its expression in rather involved and will not be reported here. It is nonetheless interesting to note that in the BCS limit

$$\gamma \simeq -\frac{256}{135}(k_{\mathrm{F}}\xi_{\mathrm{pair}})^2 \tag{73}$$

is negative (and large), so that the dispersion relation

$$\omega(\mathbf{q})^2 \simeq \frac{v_{\mathrm{F}}^2\mathbf{q}^2}{3}\left[1 - \frac{1}{3}\mathbf{q}^2\xi_{\mathrm{pair}}^2\right] \tag{74}$$

holds for $|\mathbf{q}|$ smaller then a critical value $q_{\mathrm{c}} \propto 1/\xi_{\mathrm{pair}}$. This implies that in the BCS limit the wavelength of the collective mode associated with the symmetry breaking cannot be smaller than the size of a Cooper pair.

The above results hold in three dimensions. In two dimensions both ξ_{phase} and s can be expressed in terms of elementary integrals, as shown in Appendix B.

4 Concluding remarks

In this paper we have provided the analytic solution of the crossover problem from BCS to BE in the three-dimensional case for a system of fermions interacting *via* an attractive contact interaction in free space, at the mean-field level and with the inclusion of Gaussian fluctuations. Although the assumptions required to obtain our analytic solution might be oversimplified in applications to realistic systems, it could be interesting yet to compare our analytic solution with numerical calculations describing more realistic cases. In particular, besides adopting a more sensible momentum-dependent form of the interaction potential, the free one-particle dispersion relation ought to be replaced by the actual band structure of the medium.

In addition, a detailed description of the crossover problem from BCS to BE would require one to introduce already at the mean-field level the coupling with the charge degrees of freedom, whose effects are expected to be especially important in the crossover region of interest, intermediate between these two limits [20].

In spite of these limitations, and considering the fact that analytic results for the crossover problem from BCS to BE were thus far limited to the two-dimensional case or to the two limits, our analytic solution is useful as it enables one to describe the crossover region in a compact way with very limited effort.

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Appendix A: Relevant properties of elliptic integrals

In this Appendix we briefly review the properties of elliptic integrals that are relevant to our treatment.

The elliptic integrals of the first and second kinds with modulus κ are defined by [16,21]

$$F(\alpha,\kappa) = \int_0^\alpha \mathrm{d}\varphi \frac{1}{\sqrt{1-\kappa^2\sin^2\varphi}} \tag{75}$$

$$E(\alpha,\kappa) = \int_0^\alpha \mathrm{d}\varphi \sqrt{1-\kappa^2\sin^2\varphi} \tag{76}$$

with $\kappa^2 < 1$. They satisfy the properties

$$F(n\pi,\kappa) = 2nF(\frac{\pi}{2},\kappa) \tag{77}$$

$$E(n\pi,\kappa) = 2nE(\frac{\pi}{2},\kappa) \tag{78}$$

(n integer), and are said to be *complete* when $\alpha = \pi/2$. The complete elliptic integrals admit the following series representations:

$$F(\frac{\pi}{2},\kappa) = \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2 \kappa^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \kappa^4 + \ldots + \left(\frac{(2n-1)!!}{2^n n!}\right)^2 \kappa^{2n} + \ldots\right\} \quad (79)$$

$$E(\frac{\pi}{2},\kappa) = \frac{\pi}{2}\left\{1 - \left(\frac{1}{2}\right)^2 \kappa^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{\kappa^4}{3} - \ldots - \left(\frac{(2n-1)!!}{2^n n!}\right)^2 \frac{\kappa^{2n}}{2n-1} - \ldots\right\} \quad (80)$$

$$F(\frac{\pi}{2},\kappa) = \ln\frac{4}{\kappa'} + \left(\frac{1}{2}\right)^2\left(\ln\frac{4}{\kappa'} - \frac{2}{1\cdot 2}\right)\kappa'^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\ln\frac{4}{\kappa'} - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4}\right)\kappa'^4 + \ldots \quad (81)$$

$$E(\frac{\pi}{2},\kappa) = 1 + \frac{1}{2}\left(\ln\frac{4}{\kappa'} - \frac{1}{1\cdot 2}\right)\kappa'^2 + \frac{1^2\cdot 3}{2^2\cdot 4}\left(\ln\frac{4}{\kappa'} - \frac{2}{1\cdot 2} - \frac{1}{3\cdot 4}\right)\kappa'^4 + \ldots \quad (82)$$

where $\kappa' = \sqrt{1-\kappa^2}$ is known as the complementary modulus. The representations (79, 80) are to be preferred when $\kappa^2 \ll 1$; when $\kappa^2 \simeq 1$ and $\kappa'^2 \ll 1$ the representations (81, 82) are to be preferred instead.

Equations (25, 27) of the text are obtained by adapting tabulated results and using the properties (77, 78). We obtain [22]:

$$\int_0^\infty \mathrm{d}x \frac{1}{\sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{1}{2a}F(\pi,\kappa) = \frac{1}{a}F(\frac{\pi}{2},\kappa) \quad (83)$$

$$\int_0^\infty \mathrm{d}x \frac{x^2}{\sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{aE(\pi,\kappa)}{2(a^4-b^4)} - \frac{F(\pi,\kappa)}{4a(a^2-b^2)} = \frac{a}{a^4-b^4}E(\frac{\pi}{2},\kappa) - \frac{1}{2a(a^2-b^2)}F(\frac{\pi}{2},\kappa) \quad (84)$$

where $a^2 > b^2 > -\infty$, $a^2 > 0$, and $\kappa^2 = (a^2-b^2)/(2a^2)$. Comparison of equations (83, 84) with equations (25, 27) leads us to identify

$$b^2 = -x_0,\ a^4 = 1 + x_0^2,\ \kappa^2 = \frac{\sqrt{1+x_0^2} + x_0}{2\sqrt{1+x_0^2}} \quad (85)$$

and results eventually into the right-hand sides of equations (25, 27).

Appendix B: Two-dimensional case

The analytic solution for the two-dimensional case has been already provided in reference [15]. Its derivation is simpler than for the three-dimensional counterpart obtained in this paper, since it can be expressed in terms of elementary integrals. For the sake of comparison, we report in this Appendix the two-dimensional solution on equal footing of the three-dimensional solution discussed in the text.

Quite generally, in two dimensions the bound-state energy ϵ_0 exists for any value of the interaction strength g. For the contact potential we are considering, however, the bound-state equation

$$-\frac{1}{g} = \frac{1}{\Omega}\sum_{\mathbf{k}} \frac{1}{\mathbf{k}^2/2 + \epsilon_0} = \frac{m}{2\pi}\int_0^\infty \mathrm{d}y \frac{1}{2y + \epsilon_0/\Delta} \quad (86)$$

(with $y = k^2/(2m\Delta)$) needs to be suitably regularized by introducing an ultraviolet cutoff. This cutoff can, in turn, be removed from further consideration by combining equation (86) with the gap equation (2), namely,

$$-\frac{1}{g} = \frac{1}{2\Omega}\sum_{\mathbf{k}} \frac{1}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}} = \frac{m}{4\pi}\int_{-x_0}^\infty \mathrm{d}z \frac{1}{(1+z^2)^{1/2}} \quad (87)$$

(with $z = y - x_0$ and x_0 given by Eq. (7)) which requires an analogous regularization. Performing the elementary integrations in equations (86, 87) one obtains

$$\frac{\epsilon_0}{\Delta} = \sqrt{1+x_0^2} - x_0. \quad (88)$$

The normalization condition (3) gives further

$$n = \frac{m\Delta}{2\pi}\int_{-x_0}^\infty \mathrm{d}z \left(1 - \frac{z}{\sqrt{1+z^2}}\right) = \frac{m\Delta}{2\pi}\left(x_0 + \sqrt{1+x_0^2}\right). \quad (89)$$

Since in the normal state $n = k_{\mathrm{F}}^2/(2\pi) = m\,\epsilon_{\mathrm{F}}/\pi$, equation (89) reads

$$\frac{\Delta}{\epsilon_{\mathrm{F}}} = \frac{2}{x_0 + \sqrt{1+x_0^2}}\,. \quad (90)$$

Multiplying at this point both sides of equations (88, 90) yields

$$\frac{\epsilon_0}{\epsilon_{\mathrm{F}}} = 2\frac{\sqrt{1+x_0^2} - x_0}{\sqrt{1+x_0^2} + x_0}\,. \quad (91)$$

Finally, the pair-correlation length can be obtained from its definition (*cf.* Eq. (14) for the three-dimensional case):

$$\xi_{\mathrm{pair}}^2 = \frac{1}{m^2}\frac{\int_0^\infty \mathrm{d}k (k^3\xi_k^2/E_k^6)}{\int_0^\infty \mathrm{d}k (k/E_k^2)} = \frac{2}{m\Delta}\frac{\int_0^\infty \mathrm{d}y (y\xi_y^2/E_y^6)}{\int_0^\infty \mathrm{d}y (1/E_y^2)} \quad (92)$$

with the dimensionless quantities

$$\begin{cases} y = k^2/(2m\Delta), \ x_0 = \mu/\Delta, \\ \xi_y = y - x_0, \qquad E_y = \sqrt{\xi_y^2 + 1} \end{cases} \tag{93}$$

(in the place of (7) for the three-dimensional case). The integrals in equation (92) are again elementary and give:

$$(k_{\mathrm{F}}\xi_{\mathrm{pair}})^2 = = \frac{1}{2}\frac{\epsilon_{\mathrm{F}}}{\Delta}\left[x_0 + \left(\frac{2+x_0^2}{1+x_0^2}\right)\left(\frac{\pi}{2} + \arctan x_0\right)^{-1}\right]. \tag{94}$$

It is then clear that $\Delta/\epsilon_{\mathrm{F}}$, $\mu/\epsilon_{\mathrm{F}} = x_0\Delta/\epsilon_{\mathrm{F}}$, $\epsilon_0/\epsilon_{\mathrm{F}}$, and $k_{\mathrm{F}}\xi_{\mathrm{pair}}$ can be expressed as functions of x_0. (Note that no reference to the scattering amplitude a_{s} has been given in two dimensions.)

Alternatively, equation (94) with $\epsilon_{\mathrm{F}}/\Delta$ given by equation (90) can be inverted to express x_0 (as well as all other quantities) as a function of $k_{\mathrm{F}}\xi_{\mathrm{pair}}$.

At the Gaussian level, all integrals entering the definitions (3) and (3) of the coefficients of the expansions (43) and (44) are elementary in *two* dimensions. We obtain:

$$a_0 = \frac{m}{8\pi}\frac{\sqrt{1+x_0^2}+x_0}{\sqrt{1+x_0^2}} \tag{95}$$

$$a_1 = -\frac{m}{8\pi}\frac{1}{\Delta\sqrt{1+x_0^2}} \tag{96}$$

$$a_2 + b_2 = \frac{1}{48\pi}\frac{1}{\Delta}\left\{x_0 + \frac{x_0^4 + 3x_0^2 + 1}{(1+x_0^2)^{3/2}}\right\} \tag{97}$$

$$a_2 - b_2 = \frac{1}{16\pi}\frac{\sqrt{1+x_0^2}+x_0}{\Delta} \tag{98}$$

$$b_3 - a_3 = \frac{m}{16\pi}\frac{1}{\Delta^2}\frac{\sqrt{1+x_0^2}+x_0}{\sqrt{1+x_0^2}}. \tag{99}$$

These results give

$$(k_{\mathrm{F}}\xi_{\mathrm{phase}})^2 = \frac{1}{6}\frac{\epsilon_{\mathrm{F}}}{\Delta}\frac{\sqrt{1+x_0^2}}{\sqrt{1+x_0^2}+x_0}\left\{x_0 + \frac{x_0^4+3x_0^2+1}{(1+x_0^2)^{3/2}}\right\} \tag{100}$$

and

$$\left(\frac{s}{v_{\mathrm{F}}}\right)^2 = \frac{1}{4}\frac{\Delta}{\epsilon_{\mathrm{F}}}\left(\sqrt{1+x_0^2}+x_0\right). \tag{101}$$

Note that, owing to equation (90), $(s/v_{\mathrm{F}})^2 = 1/2$ is independent from x_0. On the other hand, for $k_{\mathrm{F}}\xi_{\mathrm{phase}}$ we obtain in the two limits:

$$k_{\mathrm{F}}\xi_{\mathrm{phase}} \simeq \begin{cases} x_0/\sqrt{6} & \text{BCS limit} \\ 1/\sqrt{8} & \text{BE limit,} \end{cases} \tag{102}$$

thus confirming that the BE limit depends markedly on dimensionality and shows a peculiar behavior in two dimensions [12]. Note, however, that in the BE limit the product $s\,\xi_{\mathrm{phase}}$ still coincides with the Bogoliubov result $(4m)^{-1}$.

References

1. D.S. Marshall, *et al.*, Phys. Rev. Lett. **76**, 4841 (1996).
2. A.G. Loeser, *et al.*, Science **273**, 325 (1996).
3. H. Ding, *et al.*, Nature **382**, 51 (1996).
4. Y.J. Uemura, *et al.*, Phys. Rev. Lett. **66**, 2665 (1991); Nature (London) **352**, 605 (1991).
5. P. Nozières, S. Schmitt-Rink, J. Low. Temp. Phys. **59**, 195 (1985).
6. D.M. Eagles, Phys. Rev. **186**, 456 (1969).
7. A.J. Leggett, in *Modern Trends in the Theory*, edited by A. Pekalski, J. Przystawa (Berlin, Heidelberg, NewYork: Springer, 1980), p. 13.
8. R. Friedberg, T.D. Lee, Phys. Rev. B **40** 6745 (1989).
9. M. Drechsler, W. Zwerger, Ann. Phys. (Germany) **1**, 15 (1992).
10. C.A.R. Sá de Melo, M. Randeria, J.R. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993), and references quoted therein.
11. R. Haussmann, Z. Phys. B **91**, 291 (1993); Phys. Rev. B **49**, 12975 (1994).
12. F. Pistolesi, G.C. Strinati, Phys. Rev. B **49**, 6356 (1994); B **53**, 15168 (1996).
13. J. Ranninger, J.M. Robin, M. Eschrig, Phys. Rev. Lett. **74** 4027 (1995) and references quoted therein.
14. S. Stintzing, W. Zwerger, Cond-Mat/9703129 (unpublished).
15. M. Randeria, J.-M. Duan, L.-Y. Shieh, Phys. Rev. B **41**, 327 (1990).
16. See, *e.g.*, *Handbook of Mathematical Functions*, edited by M. Abramowitz, I.A. Stegun (New York: Dover, 1972) chapter 17.
17. Cf., *e.g.*, A.L. Fetter, J.D. Walecka, *Quantum Theory of Many-Particle Systems* (New York: McGraw Hill 1971), chapter 13.
18. Cf., *e.g.*, A. Erdelyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Trascendental Functions*, chapter XIII. (New York: McGraw Hill, 1953).
19. V.N. Popov, *Functional Integrals and Collective Excitations* (Cambridge: Cambridge University Press, 1987), chapter 13.
20. T. Kostyrko, R. Micnas, Phys. Rev. B **46**, 11025 (1992).
21. I.S. Gradshteyn, I.M. Ryzhik, *Tables of Integrals, Series, and Products* (New York: Academic Press 1965) section 8.1.
22. *Cf.* Ref. [21], p. 260.