Evolution from BCS superconductivity to Bose condensation: Calculation of the zero-temperature phase coherence length

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We consider a fermionic system at zero temperature interacting through an effective nonretarded potential of the type introduced by Nozières and Schmitt-Rink, and calculate the *phase* coherence length ξ_{phase} (associated with the spatial fluctuations of the superconducting order parameter) by exploiting a functional-integral formulation for the correlation functions and the associated loop expansion. This formulation is especially suited to follow the evolution of the fermionic system from a BCS-type superconductor for weak coupling to a Bose-condensed system for strong coupling, since in the latter limit a *direct* mapping of the original fermionic system onto an effective system of bosons with a residual boson-boson interaction can be established. Explicit calculations are performed at the one-loop order. The phase coherence length ξ_{phase} is compared with the coherence length ξ_{pair} for two-electron correlation, which is relevant to distinguish the weak- ($k_F\xi_{\mathrm{pair}} \gg 1$) from the strong- ($k_F\xi_{\mathrm{pair}} \ll 1$) coupling limits ($k_F$ being the Fermi wave vector) *as well as* to follow the crossover in between. It is shown that ξ_{phase} coincides with ξ_{pair} down to $k_F\xi_{\mathrm{pair}} \simeq 10$, ξ_{pair} in turn coinciding with the Pippard coherence length. In the strong-coupling limit we find instead that $\xi_{\mathrm{phase}} \gg \xi_{\mathrm{pair}}$, with ξ_{pair} coinciding with the radius of the bound-electron pair. From the mapping onto an effective system of bosons in the strong-coupling limit we further relate ξ_{pair} with the ''range'' of the residual boson-boson interaction, which is physically the only significant length associated with the dynamics of the bosonic system. [S0163-1829(96)02422-8]

I. INTRODUCTION

There has been recently renewed interest in the crossover from BCS superconductivity to Bose-Einstein (BE) condensation, following the discovery of the high-temperature superconductors.[1–8] In particular, the observation that these (as well as other ''exotic'') superconductors have considerably (i.e., 10^3–10^4 times) shorter *coherence length* than conventional superconductors has prompted the suggestion that proper description of superconductivity in these materials might require an *intermediate* approach between the two limits represented by BCS theory and BE condensation.[9] In this context, it appears especially relevant to assess how the coherence length (which can be determined experimentally from the spatial fluctuations of the order parameter and which we shall consistently refer to as ξ_{phase} in the following) crosses over between these two limits. The purpose of this paper is to provide a detailed description of this crossover.[10]

Evolution from weak- to strong-coupling superconductivity was addressed a few years ago by Nozières and Schmitt-Rink[11] (hereafter referred to as NSR) after the pioneering work by Leggett.[12] NSR follow this evolution by increasing the coupling strength of an effective fermionic attractive potential, and conclude that the evolution is ''smooth.'' The inclusion of fluctuations beyond mean field considered by NSR through the ladder approximation for the pairing susceptibility, however, has posed problems of physical consistency,[13] owing to the fact that the ladder approximation is not ''conserving.''[14] This shortcoming was later overcome by Haussmann[15] who considered a fully ''conserving'' diagrammatic approach to describe the interacting Fermi system in the superconducting phase, whereby each single-particle Green's function is self-consistently determined. It turns out that keeping the full self-consistency is most important in the intermediate (crossover) region of interest, in order to account correctly for the mixture of fermionic and bosonic degrees of freedom.[15]

The approaches of Refs. 13 and 15 (as well as the related work of Refs. 1–8) rely on an approximation scheme (i.e., BCS mean field plus fluctuations) which is well established in the weak-coupling limit. The fact that this procedure results in a sensible strong-coupling limit (i.e., the noninteracting Bose gas of Ref. 13 or the weakly interacting Bose gas of Ref. 15) can be related to the structure of the BCS wave function, which has built in the BE condensation as a limiting case.[16] There is, however, *a priori* no guarantee that the results in the strong-coupling limit would always provide a satisfactory description of the limiting system of interacting bosons.

For these reasons, we prefer to approach the bosonization process *in reverse*, that is, by setting up first a reliable approximation for the bosonic system and then determining how the bosonization procedure of the original fermionic system maps that approximation back onto a description of the weak-coupling limit. In this way, we can focus directly on improving the description of the bosonic limit, which is admittedly more difficult to deal with than the opposite weak-coupling limit, where the BCS approximation is expected to be invariably recovered as the fundamental starting point.

Focusing directly on the bosonic limit, however, leads us to confront a long-standing problem in the theory of interacting bosons. It has, in fact, long been known that conventional many-body (diagrammatic) methods for an interacting *condensed* Bose system can be quite generally organized within approximation schemes that are consistent *either* with conservation laws (''Φ-derivable'' approximations) *or* with the absence of a gap in the elementary excitations spectrum (''gapless'' approximations).[17] This difficulty does not appear in the corresponding scheme for self-consistent ''Φ-derivable'' (conserving) approximations for fermionic systems (*even* in the superconducting phase).[18] For these reasons, when dealing with condensed bosonic systems one prefers to abandon self-consistent schemes and resorts instead to approximation procedures whereby diagrams are selected in terms of an *external* small parameter (like the reduced density).[19]

An approach formally alternative to conventional diagrammatic methods to set up a modified perturbation theory for a superfluid Bose system is the functional-integral method with the associated ''loop'' expansion, which allows for a unified description of superfluidity and superconductivity in terms of collective variables.[20] By this method, bosonic-like collective variables are introduced at the outset in the description of the fermionic superconducting system of interest via a Hubbard-Stratonovich transformation, in terms of which a mean-field approximation and the associated fluctuation corrections can be defined. Specifically, the mean-field approximation recovers the NSR results obtained at zero temperature, while systematic inclusion of fluctuation corrections by the loop expansion enables one to overcome the problems of physical consistency mentioned above for the NSR results.[21] In this context, it is worth mentioning the recent work by Traven[22] who considered the interaction between pair fluctuations (which are ignored by the standard Gaussian approximation) and demonstrated that it removes the pathological behavior of the thermodynamic functions obtained within the Gaussian approximation in two dimensions, thus stabilizing the low-temperature superfluid phase. We have also to mention in this context that the loop expansion associated with the functional integral can be formally mapped[23] in the bosonic limit onto the low-density expansion (which is conventionally used to select the relevant diagrammatic structure for the *dilute* Bose gas[19]). Keeping all terms up to a given order in the expansion parameter further guarantees that conservation laws *and* Ward identities are satisfied up to the same order. It is in this sense that the problems originating from the ''gapless'' and ''Φ-derivable'' approximations are overcome by the ''loop'' expansion. In the following, we shall apply the functional-integral method at the one-loop (i.e., the next-to-significant) order to the problem of the crossover between BCS and BE, with the same-model Hamiltonian adopted by NSR.

Returning, specifically, to the calculation of the phase coherence length $\xi_{\rm phase}$ at zero temperature, we will show that the one-loop calculation leads to a consistent picture for the crossover of this physical quantity, which varies from the Pippard coherence length ξ_0 in the weak-coupling limit to the known result $(4m_B\mu_B)^{-1/2}$ for a dilute Bose gas (with mass m_B and chemical potential μ_B) in the strong-coupling limit.[19] These results will be contrasted with the (mean-field) calculation of the coherence length $\xi_{\rm pair}$ for two-electron correlation reported previously,[24] which ranges instead from ξ_0 to the bound-state radius r_0 in the two limits. In Ref. 24 it was also concluded that (i) $\xi_{\rm pair}$ (through the dimensionless parameter $k_F\xi_{\rm pair}$) is the relevant variable to *follow* the crossover from BCS to BE [and thus it does not serve to identify merely the two extreme BCS ($k_F\xi_{\rm pair}\gg 1$) and BE ($k_F\xi_{\rm pair}\ll 1$) limits], and (ii) this crossover occurs in practice in a limited range of the variable $k_F\xi_{\rm pair}$ (beginning at $k_F\xi_{\rm pair}\simeq 10$ on the BCS side). The calculation of $\xi_{\rm phase}$ reported here confirms this result, because we will find that $\xi_{\rm phase}\simeq\xi_{\rm pair}$ down to $k_F\xi_{\rm pair}\simeq 10$, with the two lengths starting to differentiate for smaller values of $k_F\xi_{\rm pair}$.[25] One can thus associate a *single* characteristic length to a BCS-type superconductor, which is generically identified (even for relatively strong coupling) by the existence of a well-defined Fermi surface. In the bosonic limit, we will find instead that $\xi_{\rm phase}\gg\xi_{\rm pair}$, as expected, since the ''size'' of a single boson is by no means related to the range of the fluctuations of the order parameter.[26] We shall further show in this limit that $\xi_{\rm pair}$ is associated with the range of the *residual* boson-boson interaction, by mapping the original fermionic system onto an effective system of interacting bosons. In this way, a sensible and consistent description of the bosonization process results from our one-loop calculation, at least in the zero-temperature limit we are considering.

The plan of the paper is the following. In Sec. II we set up the calculation of $\xi_{\rm phase}$ at the one-loop order, by relying on a functional-integral representation of the correlation functions for a fermionic system interacting through an effective potential of the type introduced by NSR. We provide also analytic expressions of $\xi_{\rm phase}$ in the weak- and strong-coupling limits. In Sec. III we consider specifically the strong-coupling limit and perform a mapping of the effective action of the original fermionic system onto the corresponding action of a truly bosonic system, by exploiting features of the collective bosonic-like variables introduced in Sec. II via a Hubbard-Stratonovich transformation. In Sec. IV we present numerical results for $\xi_{\rm phase}$ (in three and lower dimensions) over the whole range of coupling, and especially across the *narrow* region of the variable $k_F\xi_{\rm pair}$ where the actual crossover from BCS to BE takes place. Section V gives our conclusions. Details of the calculations as well as related additional material are given in the Appendixes. In particular, Appendix A obtains the shift of the order parameter which is required to make the NSR approach fully consistent at the one-loop level in the condensed phase (or in two dimensions[22]). In Appendix D the ''universal'' curve, obtained previously in Ref. 24 for the chemical potential versus $k_F\xi_{\rm pair}$ using the NSR separable interaction, is discussed further in the context of the (three-dimensional) negative-U Hubbard model and of the analytic two-dimensional solution of Ref. 2.

II. CALCULATION OF THE ZERO-TEMPERATURE PHASE COHERENCE LENGTH FOR A SUPERCONDUCTING FERMIONIC SYSTEM

In this section we consider the calculation of the spatial fluctuations of the order parameter $\langle\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})\rangle$ for a superconducting fermionic system [$\psi_\sigma(\mathbf{r})$ being the fermionic field operator with spin projection σ], and follow their evolution as the fermionic system is driven toward its bosonic counterpart by increasing the (effective) attractive fermionic interaction. To this end (and for the reasons discussed in the Introduction), we shall rely on a functional-integral representation of the correlation functions and the associated loop expansion. Since some confusion has sometimes arisen in the literature between the inclusion of Gaussian fluctuations and the consistency of the loop expansion,[7,13] we shall discuss the latter in some detail in the following (see also Appendix A).

To identify the range of the spatial fluctuations of the superconducting order parameter across the evolution from BCS to BE, we find it convenient to introduce the bosonic-type operator

$$\varphi(\mathbf{R}) = \int d\boldsymbol{\rho}\, \phi(2\boldsymbol{\rho})\, \psi_\uparrow(\mathbf{R}-\boldsymbol{\rho})\, \psi_\downarrow(\mathbf{R}+\boldsymbol{\rho}), \tag{2.1}$$

where the (real) function $\phi(\boldsymbol{\rho})$ is assumed to be ''localized'' about $\boldsymbol{\rho}=0$. The thermal average $\langle\varphi(\mathbf{R})\rangle$ can be then associated with the order parameter of the broken-symmetry phase. Since this order parameter is (in general) complex, we can further represent the operator (2.1) via its *longitudinal* and *transverse* components to the direction of broken symmetry:[27]

$$\varphi_\parallel(\mathbf{R}) = \frac{1}{2|\Delta|}\,[\Delta^*\varphi(\mathbf{R}) + \Delta\varphi^\dagger(\mathbf{R})], \tag{2.2a}$$

$$\varphi_\perp(\mathbf{R}) = \frac{1}{2i|\Delta|}\,[\Delta^*\varphi(\mathbf{R}) - \Delta\varphi^\dagger(\mathbf{R})], \tag{2.2b}$$

where we have set

$$\Delta = \langle \varphi(\mathbf{R})\rangle. \tag{2.3}$$

The relevant correlation functions for the operators (2.2) can then be defined as follows:

$$F^\parallel(\mathbf{R}-\mathbf{R}') = \int_0^\beta d\tau \langle T_\tau[\varphi_\parallel(\mathbf{R},\tau)\varphi_\parallel(\mathbf{R}',\tau=0)]\rangle - \beta|\Delta|^2, \tag{2.4a}$$

$$F^\perp(\mathbf{R}-\mathbf{R}') = \int_0^\beta d\tau \langle T_\tau[\varphi_\perp(\mathbf{R},\tau)\varphi_\perp(\mathbf{R}',\tau=0)]\rangle, \tag{2.4b}$$

where T_τ is the imaginary-time T-ordering operator, the thermal average $\langle\cdots\rangle$ is taken at the equilibrium temperature β^{-1}, and the Heisenberg representation for the field operators is implemented below. Note that the unnecessary dependence on the imaginary time τ has been eliminated in the expressions (2.4) by the time averaging.

For a homogeneous system, it is further convenient to introduce the Fourier transform of the operator (2.1):

$$\varphi(\mathbf{q}) = \frac{1}{\sqrt{\Omega}} \int d\mathbf{R}\, e^{-i\mathbf{q}\cdot\mathbf{R}} \varphi(\mathbf{R}) = \sum_{\mathbf{k}} \phi(\mathbf{k})\, c_\uparrow(\mathbf{k}+\mathbf{q}/2)\, c_\downarrow(-\mathbf{k}+\mathbf{q}/2), \tag{2.5}$$

where $\phi(\mathbf{k})$ is the Fourier transform of the function $\phi(\boldsymbol{\rho})$ of Eq. (2.1), $c_\sigma(\mathbf{k})$ is the destruction operator of wave vector $\mathbf{k}$ and spin σ, and Ω is the volume occupied by the system. In terms of the operator (2.5) we rewrite

$$\begin{aligned} F^{\parallel,\perp}(\mathbf{R}-\mathbf{R}') = \pm\frac{1}{4\Omega}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot(\mathbf{R}-\mathbf{R}')}\int_0^\beta d\tau \{ &\langle T_\tau[\varphi(\mathbf{q},\tau)\varphi(-\mathbf{q},\tau=0)]\rangle \pm \langle T_\tau[\varphi(\mathbf{q},\tau)\varphi(\mathbf{q},\tau=0)^\dagger]\rangle \\ &\pm \langle T_\tau[\varphi(-\mathbf{q},\tau)^\dagger\varphi(-\mathbf{q},\tau=0)]\rangle + \langle T_\tau[\varphi(-\mathbf{q},\tau)^\dagger\varphi(\mathbf{q},\tau=0)^\dagger]\rangle\} - \frac{(1\pm1)}{2}\beta|\Delta|^2, \end{aligned} \tag{2.6}$$

where the upper (lower) sign refers to $F^\parallel$ ($F^\perp$). Note that the τ-averaging selects the zero- (Matsubara) frequency component of the correlation functions within braces in Eq. (2.6). Note also that in Eq. (2.6) we have eventually considered Δ to be real.

Below a critical temperature, one expects to identify a *finite* coherence length for longitudinal correlations only. In particular, the behavior for *small* $\mathbf{q}$ of the integrand in Eq. (2.6) is of interest whenever the correlation function $F^\parallel(\mathbf{R}-\mathbf{R}')$ has a well-behaved (exponential) spatial decay. Since the broken-symmetry condition resides in the *phase* of the order parameter (2.3), in the following we shall identify as ξ_{phase} the coherence length associated with $F^\parallel$. Physically, ξ_{phase} provides an estimate of the spatial dimension over which the phase fluctuations are correlated. In the strong-coupling (BE) limit one thus expects ξ_{phase} to be much larger than the typical size of the fermionic pairs (which, in this limit, constitute truly bosonic entities). In this context, it is relevant to introduce an additional length (say, ξ_{pair}) which reduces to the size of the bound fermionic *pair* in the BE limit. On general ground, information on ξ_{pair} can be extracted from the fermionic pair-correlation function (with opposite spins)

$$g(\mathbf{r}) = \frac{1}{n^2}\langle \psi_\uparrow^\dagger(\mathbf{r})\psi_\downarrow^\dagger(0)\psi_\downarrow(0)\psi_\uparrow(\mathbf{r})\rangle - \frac{1}{4}, \tag{2.7}$$

where n is the particle density and the constant Hartree term has been subtracted for convenience. For instance, at the mean-field level Eq. (2.7) becomes

$$g(\mathbf{r}) = \frac{1}{n^2}\,|\langle\Phi|\psi_\uparrow^\dagger(\mathbf{r})\psi_\downarrow^\dagger(0)|\Phi\rangle|^2, \tag{2.8}$$

where $|\Phi\rangle$ is the BCS ground state, and ξ_{pair} can be obtained as[24]

$$\xi_{\mathrm{pair}}^2=\frac{\int d\mathbf{r}\, g(\mathbf{r})\mathbf{r}^2}{\int d\mathbf{r}\, g(\mathbf{r})}. \tag{2.9}$$

In the BE limit, ξ_{pair} obtained from Eq. (2.9) coincides with the bound-state radius of the associated two-fermion problem: At the mean-field level a *single* length enters the function (2.8) since no correlation is established between bound pairs. Beyond the mean field, however, correlation between bound pairs should occur and the length ξ_{phase} should affect $g(\mathbf{r})$. Nonetheless, in the BE limit we expect the magnitudes of the two lengths ξ_{pair} and ξ_{phase} to be widely separated, in such a way that ξ_{pair} can still be extracted from $g(\mathbf{r})$ by inspection. For this reason, in the following we shall restrict in practice to the mean-field definition (2.9) with $g(\mathbf{r})$ given by Eq. (2.8). In the weak-coupling limit, on the other hand, we expect no difference between ξ_{phase} and ξ_{pair} (apart, possibly, from a trivial normalization factor due to the respective definitions). In other words, in the weak-coupling limit a *single* length characterizes the correlation within a Cooper pair *and* among different Cooper pairs (the correlation originating essentially from Pauli exclusion principle).

To proceed in the calculation of ξ_{phase} (and ξ_{pair}) we need a specific Hamiltonian to describe the interacting fermionic system. To connect with previous work on the crossover from BCS to BE, we adopt the model Hamiltonian considered by NSR:

$$H=\sum_{\mathbf{k},\sigma}\xi_{\mathbf{k}}c_\sigma^\dagger(\mathbf{k})c_\sigma(\mathbf{k})+\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}V(\mathbf{k},\mathbf{k}')c_\uparrow^\dagger(\mathbf{k}+\mathbf{q}/2)\times c_\downarrow^\dagger(-\mathbf{k}+\mathbf{q}/2)c_\downarrow(-\mathbf{k}'+\mathbf{q}/2)c_\uparrow(\mathbf{k}'+\mathbf{q}/2) \tag{2.10}$$

with $\xi_{\mathbf{k}}=\mathbf{k}^2/2m-\mu$ (μ being the chemical potential).[28] This Hamiltonian differs from the usual BCS reduced Hamiltonian,[16] in that it allows for finite values of the (center-of-mass) momentum $\mathbf{q}$ of the pair operator $c_\uparrow^\dagger c_\downarrow^\dagger$ while keeping the singlet spin pairing. Taking into account finite values of $\mathbf{q}$ is, in fact, necessary to represent the strong-coupling limit in terms of interacting bosons.[29]

For convenience, we also take the (effective) attractive interaction potential in Eq. (2.10) of the separable form ($V<0$):

$$V(\mathbf{k},\mathbf{k}')=Vw(\mathbf{k})w(\mathbf{k}'). \tag{2.11}$$

In the usual BCS theory, $w(\mathbf{k})=\theta(\epsilon_c-|\xi_{\mathbf{k}}|)$ specifies an abrupt cutoff about the Fermi surface (ϵ_c being the cutoff energy). To treat the strong-coupling limit on the same footing of the weak-coupling limit, $w(\mathbf{k})$ should instead interpolate smoothly between small and large $\mathbf{k}$. We take accordingly

$$w(\mathbf{k})=[1+(\mathbf{k}/k_0)^2]^{-\gamma} \tag{2.12}$$

with $\gamma>0$. We have verified that the restriction $1/4<\gamma<3/4$ ensures the relevant correlation functions to be well defined via their Fourier transforms, as well as the bound-state radius for the associated two-body problem to vanish in the limit $|V|\to\infty$. In most of the following calculations we shall take the value $\gamma=1/2$ considered by NSR.[30]

A. Functional-integral approach

As discussed in the Introduction, although we are originally considering a system of interacting fermions, we are interested in treating properly the strong-coupling regime where the fermionic system gets mapped onto a system of interacting bosons. To this end, it is relevant to introduce for *any* coupling bosonic-like (collective) variables from the outset, which turn eventually into truly bosonic fields in the strong-coupling limit.

Functional integrals are especially suited for introducing collective variables and, at the same time, for providing one with conserving approximations even in the presence of condensates.[20] In this context, one obtains the relevant fermionic correlation functions by differentiating the generating functional[31]

$$Z[\overline{\eta},\eta]=\frac{\int \mathscr{D}\overline{c}\mathscr{D}c\, \exp\{-S-S_{\mathrm{int}}\}}{\int \mathscr{D}\overline{c}\mathscr{D}c\, \exp\{-S\}} \tag{2.13}$$

with respect to the "sources" $\eta(\overline{\eta})$, where

$$S=\int_0^\beta d\tau\left(\sum_{\mathbf{k},\sigma}\overline{c_\sigma}(\mathbf{k},\tau)\,\frac{\partial}{\partial\tau}\,c_\sigma(\mathbf{k},\tau)+H(\tau)\right) \tag{2.14}$$

and

$$S_{\mathrm{int}}=\int_0^\beta d\tau\sum_{\mathbf{k},\sigma}[\overline{\eta_\sigma}(\mathbf{k},\tau)c_\sigma(\mathbf{k},\tau)+\overline{c_\sigma}(\mathbf{k},\tau)\,\eta_\sigma(\mathbf{k},\tau)]. \tag{2.15}$$

In these expressions, $(c,\overline{c})$ and $(\eta,\overline{\eta})$ are Grassmann variables, and

$$H(\tau)=\sum_{\mathbf{k},\sigma}\xi_{\mathbf{k}}\overline{c_\sigma}(\mathbf{k},\tau)c_\sigma(\mathbf{k},\tau)+V\sum_{\mathbf{q}}\overline{\mathscr{B}}(\mathbf{q},\tau)\mathscr{B}(\mathbf{q},\tau) \tag{2.16}$$

with

$$\mathscr{B}(\mathbf{q},\tau)=\sum_{\mathbf{k}}w(\mathbf{k})c_\downarrow(-\mathbf{k}+\mathbf{q}/2,\tau)c_\uparrow(\mathbf{k}+\mathbf{q}/2,\tau) \tag{2.17}$$

is the Hamiltonian associated in the action with the operator (2.10) and the choice (2.11) for the interaction potential.[32]

Bosonic-like variables are introduced via the Hubbard-Stratonovich transformation

$$\exp\{-V\overline{\mathscr{B}}(\mathbf{q},\tau)\mathscr{B}(\mathbf{q},\tau)\}=-\frac{1}{\pi V}\int db^*(\mathbf{q},\tau)db(\mathbf{q},\tau)\times\exp\left\{\frac{1}{V}\left|b(\mathbf{q},\tau)\right|^2+b(\mathbf{q},\tau)\overline{\mathscr{B}}(\mathbf{q},\tau)+b^*(\mathbf{q},\tau)\mathscr{B}(\mathbf{q},\tau)\right\} \tag{2.18}$$

that holds for any $\mathbf{q}$ and τ, where the variables $b(\mathbf{q},\tau)$ obey periodic boundary conditions $b(\mathbf{q},\tau+\beta)=b(\mathbf{q},\tau)$.[33] It is further convenient to introduce time Fourier transforms for Grassmann and bosonic variables, and make the change of variables

$$\chi_1(k)=\overline{c_\uparrow}(k), \quad \chi_2(k)=c_\downarrow(-k),$$
$$\xi_1(k)=-\overline{\eta_\uparrow}(k), \quad \xi_2(k)=\eta_\downarrow(-k), \tag{2.19}$$

analogous to Nambu spinor transformation, with the shorthand notation $k=(\mathbf{k},\omega_s)$ where $\omega_s=2\pi(s+1/2)\beta^{-1}$ (s integer) is a fermionic Matsubara frequency. The generating functional (2.13) can thus be rewritten in the form

$$Z[\overline{\xi},\xi]=\frac{\int \mathcal{D}\overline{\chi}\mathcal{D}\chi\,\mathcal{D}b^*\mathcal{D}b\,\exp\{-S'-S'_{\rm int}\}}{\int \mathcal{D}\overline{\chi}\mathcal{D}\chi\,\mathcal{D}b^*\mathcal{D}b\,\exp\{-S'\}}, \tag{2.20}$$

where now

$$S'=-\frac{1}{\beta V}\sum_q |b(q)|^2 + \sum_{k,k'} (\overline{\chi_1}(k),\overline{\chi_2}(k))\mathbf{M}(k,k')\begin{pmatrix}\chi_1(k')\\ \chi_2(k')\end{pmatrix} \tag{2.21}$$

and

$$S'_{\rm int}=\sum_k \sum_{i=1}^{2} [\overline{\xi_i}(k)\chi_i(k)+\overline{\chi_i}(k)\xi_i(k)]. \tag{2.22}$$

In Eq. (2.21), $q=(\mathbf{q},\omega_\nu)$ where $\omega_\nu=2\pi\nu\beta^{-1}$ (ν integer) is a bosonic Matsubara frequency, and $\mathbf{M}(k,k')$ is the 2×2 matrix

$$\mathbf{M}(k,k') = \begin{pmatrix} \epsilon(k)\,\delta_{k,k'}, & \frac{1}{\beta}\,w\left(\frac{k+k'}{2}\right)b^*(k-k') \\ \frac{1}{\beta}\,w\left(\frac{k+k'}{2}\right)b(k'-k), & -\epsilon(-k)\,\delta_{k,k'}\end{pmatrix} \tag{2.23}$$

with

$$\epsilon(k)=i\omega_s-\xi_{\mathbf{k}}. \tag{2.24}$$

The Grassmann variables can be integrated out at this point in Eq. (2.20), yielding

$$Z[\overline{\xi},\xi]=\frac{\int \mathcal{D}b^*\mathcal{D}b\,\exp\{-S_{\rm eff}-S''_{\rm int}\}}{\int \mathcal{D}b^*\mathcal{D}b\,\exp\{-S_{\rm eff}\}}, \tag{2.25}$$

where

$$S_{\rm eff}=-\frac{1}{\beta V}\sum_q |b(q)|^2-\mathrm{tr}\,\ln\mathbf{M} \tag{2.26}$$

is the *effective* bosonic action and

$$S''_{\rm int}=\sum_{k,k'}\sum_{i,i'=1}^{2}\overline{\xi_i}(k)\mathbf{M}^{-1}(k,k')_{i,i'}\xi_{i'}(k'). \tag{2.27}$$

Note that the trace in Eq. (2.26) is performed over the four-momentum (k) and Nambu spin (i) indices, and that $S_{\rm eff}$ formally contains all powers in the bosonic variables b.

To proceed further, one usually considers a quadratic (Gaussian) expansion of the effective action (2.26) in terms of $(b-b_0)$ where b_0 is a mean-field value. In particular, in Ref. 7 this procedure has been applied to derive the analog of the time-dependent Ginzburg-Landau equation for the crossover problem from BCS to BE above the mean-field critical temperature (where b_0 vanishes identically). For the zero-temperature properties we are interested in, however, a straightforward Gaussian expansion is known to be not fully consistent since it omits contributions that are formally of the same order of the Gaussian contributions itself (cf., e.g., Ref. 34 for the zero-temperature properties of a three-dimensional dilute Bose gas and Ref. 22 for its two-dimensional counterpart). To keep full consistency at each stage of the calculation, we introduce a (formal) *loop expansion* in the generating functional (2.25) by (i) replacing the effective action $S_{\rm eff}$ with $S_{\rm eff}/\lambda$ where $0<\lambda\leqslant1$; (ii) regarding λ as the expansion parameter of the theory (to express, e.g., the correlation functions as power series in λ); (iii) setting $\lambda=1$ eventually at the end of the calculation. In this way, expansion of the relevant physical quantities up to a given order in λ guarantees conservation laws and Ward identities to be satisfied to the same order in the expansion.[35] Note that, contrary to other cases for which a ''small'' loop parameter naturally emerges from the physics of the problem, the introduction of a loop parameter in the present context might at first look somewhat artificial. As mentioned in the Introduction, however, it can be shown that the present loop expansion gets formally mapped onto a *low-density* expansion in the bosonic limit.[23]

To implement the loop expansion, we set

$$b(q)=\beta[\Delta_0\delta_{q,0}+\sqrt{\lambda}\,\underset{\sim}{b}(q)], \tag{2.28}$$

where Δ_0 plays the role of a (complex) bosonic condensate and $\underset{\sim}{b}$ of its fluctuating part. The matrix (2.23) becomes accordingly

$$\mathbf{M}_\lambda(k,k')=\mathbf{M}_0(k,k')+\sqrt{\lambda}\,\mathbf{M}_1(k,k'), \tag{2.29}$$

where

$$\mathbf{M}_0(k,k')=\begin{pmatrix}\epsilon(k) & w(k)\Delta_0^* \\ w(k)\Delta_0 & -\epsilon(-k)\end{pmatrix}\delta_{k,k'} \tag{2.30}$$

and

$$\mathbf{M}_1(k,k')=w\left(\frac{k+k'}{2}\right)\begin{pmatrix}0 & \underset{\sim}{b}^*(k-k') \\ \underset{\sim}{b}(k'-k) & 0\end{pmatrix} \tag{2.31}$$

are independent of λ. Correspondingly, the effective action reads

$$\frac{S_{\rm eff}}{\lambda}=-\frac{\beta}{V}\left(\frac{|\Delta_0|^2}{\lambda}+\frac{\Delta_0}{\sqrt{\lambda}}\,\underset{\sim}{b}^*(q=0)+\frac{\Delta_0^*}{\sqrt{\lambda}}\,\underset{\sim}{b}(q=0)+\sum_q |\underset{\sim}{b}(q)|^2\right)-\frac{1}{\lambda}\left(\mathrm{tr}\,\ln\mathbf{M}_0-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\times\lambda^{n/2}\,\mathrm{tr}(\mathbf{M}_0^{-1}\mathbf{M}_1)^n\right). \tag{2.32}$$

The constant Δ_0 is determined, as usual, by requiring the coefficients of the linear terms in $\underset{\sim}{b}(q=0)$ and $\underset{\sim}{b}^*(q=0)$ to vanish, yielding the BCS ''gap equation''

$$\Delta_0=-V\sum_{\mathbf{k}}\frac{\Delta(\mathbf{k})w(\mathbf{k})}{2E(\mathbf{k})}\tanh\left(\frac{\beta E(\mathbf{k})}{2}\right) \tag{2.33}$$

with $\Delta(\mathbf{k})=\Delta_0 w(\mathbf{k})$ and $E(\mathbf{k})=\sqrt{\xi_{\mathbf{k}}^2+|\Delta(\mathbf{k})|^2}$.

Equation (2.32) is still exact. Approximations depend on the number of powers $\lambda^{n/2}$ considered. In particular, the Gaussian approximation for S_{eff} results upon keeping the next significant $(n=2)$ order in λ, namely, by taking

$$\frac{S_{\text{eff}}}{\lambda}=\frac{1}{\lambda}\,\beta F_0+S_{\text{eff}}^{(2)}, \tag{2.34}$$

where

$$F_0=-\frac{|\Delta_0|^2}{V}-\frac{1}{\beta}\,\text{tr}\,\ln\mathbf{M}_0 \tag{2.35}$$

is the (grand-canonical) free energy at the mean-field level[36] and $S_{\text{eff}}^{(2)}$ is the quadratic form

$$S_{\text{eff}}^{(2)}=\frac{1}{2}\sum_{q}\,(\underset{\sim}{b}^*(q),\underset{\sim}{b}(-q))\begin{pmatrix}A(q) & B(q)\\ B^*(q) & A(-q)\end{pmatrix}\begin{pmatrix}\underset{\sim}{b}(q)\\ \underset{\sim}{b}^*(-q)\end{pmatrix}. \tag{2.36}$$

In this expression

$$A(q)=-\frac{\beta}{V}-\sum_{k}\,w(k-q/2)^2\mathscr{G}(k)\mathscr{G}(q-k)=A(-q)^*, \tag{2.37}$$

$$B(q)=\sum_{k}\,w(k-q/2)^2\mathscr{F}(k)\mathscr{F}(q-k)=B(-q), \tag{2.38}$$

where

$$\mathscr{G}(k)=\mathbf{M}_0^{-1}(k)_{11}=\frac{\epsilon(-k)}{|\epsilon(k)|^2+|\Delta(k)|^2}, \tag{2.39}$$

$$\mathscr{F}(k)=\mathbf{M}_0^{-1}(k)_{21}=\frac{\Delta(k)}{|\epsilon(k)|^2+|\Delta(k)|^2}, \tag{2.40}$$

are the ordinary Gorkov functions. $A(q)+\beta/V$ and $B(q)$ represent normal and anomalous particle-particle bubbles, respectively, as depicted in Fig. 1.[37]

In what follows, it is sufficient to retain the quadratic action (2.34) only, *but* for the calculation of the shift Δ_1 of the mean-field parameter Δ_0 for which it is necessary to keep also cubic terms in the expansion (2.32) of the effective action (see Appendix A). In this way, the (grand-canonical) free energy acquires the following correction to the next significant order beyond mean field:[38]

$$F_1=\frac{1}{2\beta}\sum_{q}\,\ln(|A(q)|^2-B(q)^2), \tag{2.41}$$

where Δ_0 has been taken to be real and with the ‘‘stability’’ conditions

$$|A(q)|-B(q)>0,\quad \text{Re}A(q)+B(q)>0. \tag{2.42}$$

The chemical potential μ can be eventually eliminated in favor of the particle density n by solving $n=-(1/\Omega)\partial F/\partial\mu$ with $F=F_0+\lambda F_1$. In principle, the chemical potential need

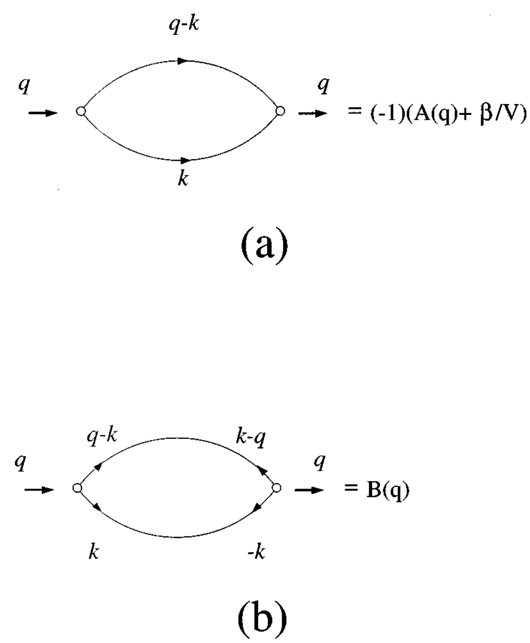

FIG. 1. Normal (a) and anomalous (b) particle-particle bubbles. Full lines represent single-particle Green's functions $[\mathscr{G}(k)=\xrightarrow{\mathbf{k}}$, $\mathscr{F}(k)=\xrightarrow{\mathbf{k}}\xleftarrow{-\mathbf{k}}]$ and empty circles represent the function $w(k)$ of the separable potential (we have assumed $-w[(k_1+k_2)/2]=\xleftarrow{\mathbf{k}_1}\!\circ\!\xrightarrow{-\mathbf{k}_2}=\xrightarrow{-\mathbf{k}_1}\!\circ\!\xleftarrow{\mathbf{k}_2}$ to be even in its argument).

not be expanded in powers of λ since all conserving requirements can be directly expressed within the grand-canonical ensemble, the mapping between μ and n being established at the end of the calculation after having set $\lambda=1$ in the expression for F. Nonetheless, one may alternatively regard μ as an internal parameter of the theory and expand it in series of λ at the outset (for details cf. Appendix C of Ref. 39). In the following, we calculate the physical quantities of interest keeping the value of μ unspecified, and expand μ in series of λ only in the final expressions.

B. Calculation of ξ_{phase} at the one-loop order

There remains to combine the calculation of the longitudinal ($F^{\parallel}$) and transverse ($F^{\perp}$) correlation functions (2.6) with the loop expansion. For completeness, we report in the following the main steps of the calculation which might also serve for addressing additional correlation functions. For our specific purposes the relevant result is Eq. (2.56) below.

We need to relate first the broken-symmetry parameter (2.3) [cf. Eq. (2.5)]

$$\Delta=\frac{1}{\sqrt{\Omega}}\,\langle\varphi(\mathbf{q}=0)\rangle=\frac{1}{\sqrt{\Omega}}\sum_{\mathbf{k}}\,\phi(\mathbf{k})\langle c_{\uparrow}(\mathbf{k})c_{\downarrow}(-\mathbf{k})\rangle \tag{2.43}$$

with the mean-field value Δ_0 and the one-loop fluctuation contribution Δ_1. To this end, we rely on the identity (proven in Appendix A)

$$\langle b(q=0)\rangle_{S_{\text{eff}}} = V\sum_k w(k)\langle c_\uparrow(k)c_\downarrow(-k)\rangle_S, \quad (2.44)$$

where the averages are taken, respectively, with actions (2.26) and (2.14). Comparison of Eq. (2.44) with the definition (2.43) then yields

$$\Delta = \frac{1}{\beta}\langle b(q=0)\rangle_{S_{\text{eff}}} = \Delta_0 + \sqrt{\lambda}\langle \underset{\sim}{b}(q=0)\rangle_{S_{\text{eff}}} \quad (2.45)$$

with the notation (2.28) and the choice $\phi(\mathbf{k}) = \sqrt{\Omega}Vw(\mathbf{k})$. The relation (2.45) is still exact. At the one-loop order it reduces to $\Delta = \Delta_0 + \lambda\Delta_1$, as shown in Appendix A.

Next, we express the averages of four-fermion operators in Eq. (2.6) [which are taken with action (2.14) within the functional-integral formulation] in terms of products of matrix elements of the inverse of the matrix (2.23) [which are correspondingly averaged with action (2.26)]. We obtain

$$F^{\parallel,\perp}(\mathbf{R}) = F_D^{\parallel,\perp}(\mathbf{R}) + F_E^{\parallel,\perp}(\mathbf{R}) - \frac{(1\pm 1)}{2}\beta|\Delta|^2, \quad (2.46)$$

where

$$\begin{aligned} F_D^{\parallel,\perp}(\mathbf{R}) = \pm\frac{V^2}{4}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{R}}\sum_{\mathbf{k},\mathbf{k}'} w(\mathbf{k})w(\mathbf{k}')\frac{1}{\beta}\sum_{s,s'} \{&\langle \mathbf{M}_{21}^{-1}(\mathbf{k}-\mathbf{q}/2,s;\mathbf{k}+\mathbf{q}/2,s)\mathbf{M}_{21}^{-1}(\mathbf{k}'+\mathbf{q}/2,s';\mathbf{k}'-\mathbf{q}/2,s')\rangle_{S_{\text{eff}}} \\ &\pm\langle \mathbf{M}_{21}^{-1}(\mathbf{k}-\mathbf{q}/2,s;\mathbf{k}+\mathbf{q}/2,s)\mathbf{M}_{12}^{-1}(\mathbf{k}'+\mathbf{q}/2,s';\mathbf{k}'-\mathbf{q}/2,s')\rangle_{S_{\text{eff}}} \\ &\pm\langle \mathbf{M}_{12}^{-1}(\mathbf{k}-\mathbf{q}/2,s;\mathbf{k}+\mathbf{q}/2,s)\mathbf{M}_{21}^{-1}(\mathbf{k}'+\mathbf{q}/2,s';\mathbf{k}'-\mathbf{q}/2,s')\rangle_{S_{\text{eff}}} \\ &+\langle \mathbf{M}_{12}^{-1}(\mathbf{k}-\mathbf{q}/2,s;\mathbf{k}+\mathbf{q}/2,s)\mathbf{M}_{12}^{-1}(\mathbf{k}'+\mathbf{q}/2,s';\mathbf{k}'-\mathbf{q}/2,s')\rangle_{S_{\text{eff}}}\} \end{aligned} \quad (2.47)$$

is the "direct" contribution, and

$$\begin{aligned} F_E^{\parallel,\perp}(\mathbf{R}) = \mp\frac{V^2}{4}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{R}}\sum_{\mathbf{k},\mathbf{k}'} w(\mathbf{k})w(\mathbf{k}')\frac{1}{\beta}\sum_{s,s'} \{&\langle \mathbf{M}_{21}^{-1}(\mathbf{k}-\mathbf{q}/2,s;\mathbf{k}'-\mathbf{q}/2,s')\mathbf{M}_{21}^{-1}(\mathbf{k}'+\mathbf{q}/2,s';\mathbf{k}+\mathbf{q}/2,s)\rangle_{S_{\text{eff}}} \\ &\pm\langle \mathbf{M}_{22}^{-1}(\mathbf{k}-\mathbf{q}/2,s;\mathbf{k}'-\mathbf{q}/2,s')\mathbf{M}_{11}^{-1}(\mathbf{k}'+\mathbf{q}/2,s';\mathbf{k}+\mathbf{q}/2,s)\rangle_{S_{\text{eff}}} \\ &\pm\langle \mathbf{M}_{11}^{-1}(\mathbf{k}-\mathbf{q}/2,s;\mathbf{k}'-\mathbf{q}/2,s')\mathbf{M}_{22}^{-1}(\mathbf{k}'+\mathbf{q}/2,s';\mathbf{k}+\mathbf{q}/2,s)\rangle_{S_{\text{eff}}} \\ &+\langle \mathbf{M}_{12}^{-1}(\mathbf{k}-\mathbf{q}/2,s;\mathbf{k}'-\mathbf{q}/2,s')\mathbf{M}_{12}^{-1}(\mathbf{k}'+\mathbf{q}/2,s';\mathbf{k}+\mathbf{q}/2,s)\rangle_{S_{\text{eff}}}\} \end{aligned} \quad (2.48)$$

is the "exchange" counterpart.

The loop expansion emerges at this point from the exact expressions (2.47) and (2.48) by interpreting the matrix $\mathbf{M}$ therein as being the matrix $\mathbf{M}_\lambda$ (2.29). In this way, its inverse acquires the expansion

$$\begin{aligned} \mathbf{M}_{\alpha\beta}^{-1} &= (\mathbf{M}_0^{-1})_{\alpha\beta} - \sqrt{\lambda}(\mathbf{M}_0^{-1}\mathbf{M}_1\mathbf{M}_0^{-1})_{\alpha\beta} \\ &\quad + \lambda(\mathbf{M}_0^{-1}\mathbf{M}_1\mathbf{M}_0^{-1}\mathbf{M}_1\mathbf{M}_0^{-1})_{\alpha\beta} + \cdots, \end{aligned} \quad (2.49)$$

yielding for the required averages

$$\begin{aligned} \langle \mathbf{M}_{\alpha\beta}^{-1}\mathbf{M}_{\gamma\delta}^{-1}\rangle &= \langle \mathbf{M}_{\alpha\beta}^{-1}\rangle\langle \mathbf{M}_{\gamma\delta}^{-1}\rangle \\ &\quad + \lambda\langle(\mathbf{M}_0^{-1}\mathbf{M}_1\mathbf{M}_0^{-1})_{\alpha\beta}(\mathbf{M}_0^{-1}\mathbf{M}_1\mathbf{M}_0^{-1})_{\gamma\delta}\rangle \\ &\quad + \mathscr{O}(\lambda^{3/2}) \end{aligned} \quad (2.50)$$

with the understanding that the product $\langle \mathbf{M}_{\alpha\beta}^{-1}\rangle\langle \mathbf{M}_{\gamma\delta}^{-1}\rangle$ is evaluated at the relevant order in λ. [In the expressions above, the indices $\alpha, \beta, \ldots$ refer to the four-vector k and the Nambu spinor component.] In particular, *at the mean-field level* Eq. (2.50) reduces to

$$\langle \mathbf{M}_{\alpha\beta}^{-1}\mathbf{M}_{\gamma\delta}^{-1}\rangle \to (\mathbf{M}_0^{-1})_{\alpha\beta}(\mathbf{M}_0^{-1})_{\gamma\delta} \quad (2.51)$$

with $\mathbf{M}_0$ given by Eq. (2.30) (and Δ_0 taken eventually to be real). In this case the "direct" contribution (2.47) becomes

$$\begin{aligned} F_D^{\parallel,\perp}(\mathbf{R}) &\to \frac{(1\pm 1)}{2}\beta\left(\frac{V}{\beta}\sum_k w(k)\mathbf{M}_0^{-1}(k)_{21}\right)^2 \\ &= \frac{(1\pm 1)}{2}\beta\Delta_0^2, \end{aligned} \quad (2.52)$$

where use has been made of the gap equation (2.33). This contribution cancels the last term on the right-hand side of Eq. (2.46) since $\Delta \to \Delta_0$ at the mean-field level. On the other hand, the "exchange" contribution (2.48) becomes

$$F_E^{\parallel,\perp}(\mathbf{R}) \to -\frac{V^2}{2\beta}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{R}}\{A(\mathbf{q},\omega_\nu=0) \pm B(\mathbf{q},\omega_\nu=0)\}, \quad (2.53)$$

where $A(q)$ and $B(q)$ are given by Eqs. (2.37) and (2.38), respectively. [Equation (2.53) holds apart from a local term proportional to the δ function of argument $\mathbf{R}$, which is consistently neglected in the following.] In particular, in the zero-temperature limit Eq. (2.53) can be cast in the form

$$F_E^{\parallel,\perp}(\mathbf{R}) \to \frac{V^2}{4}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{R}} \sum_{\mathbf{k}} \frac{w(\mathbf{k}-\mathbf{q}/2)^2}{E_{\mathbf{k}}+E_{\mathbf{k}-\mathbf{q}}} \times\left[1+\frac{\xi_{\mathbf{k}}\xi_{\mathbf{k}-\mathbf{q}} \mp \Delta_{\mathbf{k}}\Delta_{\mathbf{k}-\mathbf{q}}}{E_{\mathbf{k}}E_{\mathbf{k}-\mathbf{q}}}\right], \tag{2.54}$$

where the function to be summed over $\mathbf{q}$ is well behaved for all $\mathbf{q}$ and contributes a ''short-range'' function of $\mathbf{R}$. Expression (2.54) will be studied numerically in Sec. IV to determine its range explicitly.

The relevant ''long-range'' behavior in $\mathbf{R}$ results instead *at the one-loop level*. Entering Eq. (2.50) into Eq. (2.47), we obtain for the ''direct'' contribution

$$F_D^{\parallel,\perp}(\mathbf{R}) = \frac{(1\pm 1)}{2}\beta\Delta^2 + \lambda\frac{V^2}{2\beta}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{R}}\{A(\mathbf{q},\omega_\nu=0)\pm B(\mathbf{q},\omega_\nu=0)\} + \frac{\lambda}{2}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{R}}\frac{\beta}{A(\mathbf{q},\omega_\nu=0)\pm B(\mathbf{q},\omega_\nu=0)} \tag{2.55}$$

(apart, again, from a term proportional to a δ function of $\mathbf{R}$). Note that the first term on the right-hand side of Eq. (2.55) results from the first term on the right-hand side of Eq. (2.50) together with Eqs. (2.44) and (2.45) (cf. Appendix A). In this term Δ is meant to contain also its one-loop shift Δ_1 (which is real when Δ_0 is real), making it to cancel with the last term of Eq. (2.46). Note further that the second term on the right-hand side of Eq. (2.55) coincides formally (apart from a sign) with the mean-field contribution (2.53) once one sets $\lambda=1$, and thus shares the same ''short-range'' character. The last term on the right-hand side of Eq. (2.55), on the other hand, yields the desired ''long-range'' behavior.

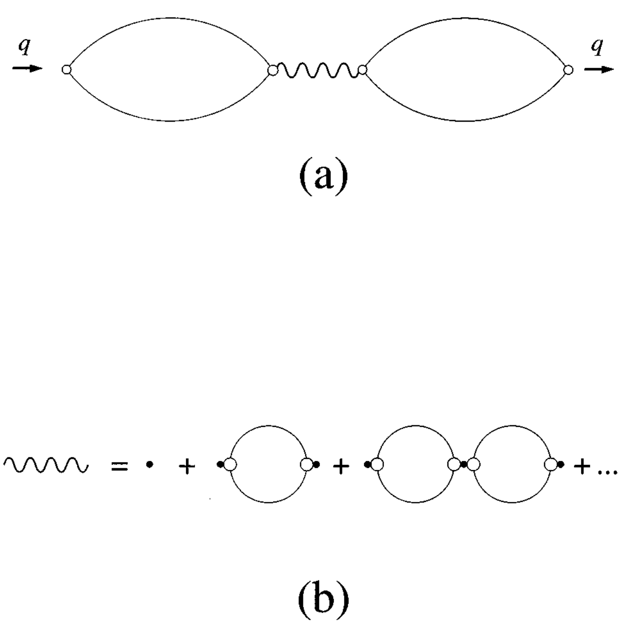

FIG. 2. (a) Graphical representation of a typical ''direct'' term of order λ in Eq. (2.47) *before* it is integrated over the wave vector $\mathbf{q}$ [recall that, by our definition (2.4) of the correlation functions, $q=(\mathbf{q},\omega_\nu=0)$ in this term]. For simplicity, arrows distinguishing normal and anomalous single-particle Green's functions are not indicated. The wiggly line stands for the (transposed) matrix of the bosonic propagator $\langle \underset{\sim}{\mathbf{b}}(q)\underset{\sim}{\mathbf{b}}^\dagger(q)\rangle_{S_{\mathrm{eff}}^{(2)}}$, where $\underset{\sim}{\mathbf{b}}(q)$ is the column vector of Eq. (2.36). This propagator is depicted in (b) (at the order considered in the present paper) as an infinite series of the original fermionic bubbles [dots represent the strength V of the separable potential (2.11)].

Before discussing this behavior in detail, it is worth representing graphically expression (2.47) at the order of the approximation (2.50). This is done in Fig. 2 for the terms of order λ. It is evident from the figure that the bosonic propagator (wiggly line) carries the external (four) momentum q, such that any singularity of this propagator for small values of q will affect the spatial decay of the ''direct'' contribution (2.47) to the correlation functions.

By contrast, in the ''exchange'' contribution (2.48) the bosonic propagator does not carry the external (four) momentum q since this propagator occurs entangled in the internal structure of the diagrams (cf. Fig. 3). In this case the singularity of the propagator for small momenta is smoothed out by the internal (four) momentum integrations in the diagrams. For this reason, the ''exchange'' diagrams are not expected to contribute to the ''long-range'' behavior of the correlation functions (2.46) and are accordingly neglected in the following.[40]

In conclusion, at the order λ (one-loop) we approximate the correlation functions (2.4) by the following expressions:

$$F^{\parallel}(\mathbf{R}) \cong \frac{\lambda}{2}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{R}}\frac{\beta}{A(\mathbf{q},\omega_\nu=0)+B(\mathbf{q},\omega_\nu=0)}, \tag{2.56a}$$

$$F^{\perp}(\mathbf{R}) \cong \frac{\lambda}{2}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{R}}\frac{\beta}{A(\mathbf{q},\omega_\nu=0)-B(\mathbf{q},\omega_\nu=0)} \tag{2.56b}$$

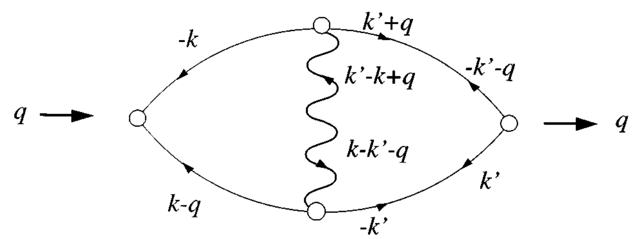

FIG. 3. Typical diagram of order λ occurring in the ''exchange'' contribution (2.48). The internal (four) momenta k and k' are integrated, and the momentum $q=(\mathbf{q},\omega_\nu=0)$ is associated with the spatial ($\mathbf{R}$) dependence. Note that diagrams of this type vanish in the ''normal'' phase (when $\Delta_0\to 0$).

with $A(q)$ and $B(q)$ given by Eqs. (2.37) and (2.38), respectively. Note that in the normal phase [for which $B(q)=0$] there is no distinction between longitudinal and transverse correlation functions. In the superconducting phase, on the other hand, a coherence length can be identified only for the longitudinal correlation function, as expected. This is because

$$\lim_{\mathbf{q}\to 0}[A(\mathbf{q},\omega_\nu=0)-B(\mathbf{q},\omega_\nu=0)]$$

$$=-\beta\left[\frac{1}{V}+\sum_{\mathbf{k}} \frac{w(\mathbf{k})^2}{2E_{\mathbf{k}}}\tanh\left(\frac{\beta E_{\mathbf{k}}}{2}\right)\right] \tag{2.57}$$

vanishes owing to the gap equation (2.33). The other combination $A(\mathbf{q},\omega_\nu=0)+B(\mathbf{q},\omega_\nu=0)$ is instead finite for $\mathbf{q}\to 0$.

If we restrict, in particular, to the *zero-temperature limit*, we obtain

$$A(\mathbf{q},\omega_\nu=0)+B(\mathbf{q},\omega_\nu=0)$$

$$=-\beta\left[\frac{1}{V}+\sum_{\mathbf{k}} w(\mathbf{k}-\mathbf{q}/2)^2\,\frac{(u_{\mathbf{k}}u_{\mathbf{k}-\mathbf{q}}-v_{\mathbf{k}}v_{\mathbf{k}-\mathbf{q}})^2}{E_{\mathbf{k}}+E_{\mathbf{k}-\mathbf{q}}}\right]$$

$$\equiv \beta f(\mathbf{q}), \tag{2.58}$$

where

$$u_{\mathbf{k}}=\sqrt{\frac{1}{2}\left(1+\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right)},\quad v_{\mathbf{k}}=\sqrt{\frac{1}{2}\left(1-\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right)} \tag{2.59}$$

are the usual BCS parameters. Note that $f(\mathbf{q})=f(|\mathbf{q}|)$ and

$$\lim_{\mathbf{q}\to 0} f(\mathbf{q})=a=\sum_{\mathbf{k}} w(\mathbf{k})^2\,\frac{\Delta_{\mathbf{k}}^2}{2E_{\mathbf{k}}^3}>0 \tag{2.60}$$

provided $\Delta_0\neq 0$, owing again to the gap equation (2.33). For small values of $\mathbf{q}$ we can thus expand

$$f(\mathbf{q})=a+b\mathbf{q}^2+\cdots, \tag{2.61}$$

and obtain the desired coherence length as follows:

$$\xi_{\text{phase}}=\sqrt{\frac{b}{a}} \tag{2.62}$$

provided b is also positive. In fact, entering the expansion (2.61) into Eq. (2.56a) yields for the leading ''long-range'' behavior (in three dimensions)

$$F^{\parallel}(\mathbf{R})\approx\frac{\lambda}{2}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{R}}\,\frac{1}{a+b\mathbf{q}^2}=\frac{\lambda}{2}\,\frac{\Omega}{4\pi b}\,\frac{\exp\{-|\mathbf{R}|/\xi_{\text{phase}}\}}{|\mathbf{R}|} \tag{2.63}$$

with ξ_{phase} given by Eq. (2.62). Consideration of the expansion (2.61) is obviously sufficient provided the function $f(\mathbf{q})$ has no other singularity.[41]

There remains to obtain an explicit expression for the coefficient b of the expansion (2.61), for generic values of the parameters characterizing the interaction potential (2.11). To this end, we expand Eq. (2.58) retaining all terms up to order $\mathbf{q}^2$ and obtain (in three dimensions)

$$\begin{aligned} b=\frac{1}{8m}\sum_{\mathbf{k}} \frac{w(\mathbf{k})^2\xi_{\mathbf{k}}^2}{E_{\mathbf{k}}^5}\Bigg\{&\frac{(\xi_{\mathbf{k}}^2-2\Delta_{\mathbf{k}}^2)}{2\xi_{\mathbf{k}}}+\frac{3}{2}z'(\mathbf{k})\Delta_{\mathbf{k}}^2\\ &+\frac{\mathbf{k}^2}{6m}\frac{\Delta_{\mathbf{k}}^2}{\xi_{\mathbf{k}}E_{\mathbf{k}}^2}[5\xi_{\mathbf{k}}+z'(\mathbf{k})(4\Delta_{\mathbf{k}}^2-6\xi_{\mathbf{k}}^2)\\ &+z'(\mathbf{k})^2\xi_{\mathbf{k}}(\xi_{\mathbf{k}}^2-4\Delta_{\mathbf{k}}^2)+3z''(\mathbf{k})\xi_{\mathbf{k}}E_{\mathbf{k}}^2]\Bigg\}\end{aligned} \tag{2.64}$$

with the notation

$$z'(\mathbf{k})=\frac{2m}{w(\mathbf{k})}\,\frac{dw(\mathbf{k})}{d\mathbf{k}^2},\quad z''(\mathbf{k})=\frac{(2m)^2}{w(\mathbf{k})}\,\frac{d^2w(\mathbf{k})}{d(\mathbf{k}^2)^2}. \tag{2.65}$$

Quite generally, all terms within braces in Eq. (2.64) contribute to the value of b for intermediate coupling and the sum over the wave vector has correspondingly to be evaluated numerically. This task will be performed in Sec. IV. In the extreme (weak- and strong-coupling) limits, on the other hand, only a single term within braces in Eq. (2.64) (albeit different in the two cases) contributes to the value of b and the sum over the wave vector can be evaluated analytically.

C. Analytic results in the BCS and BE limits

It is worth showing in detail how the coefficients a and b of the expansion (2.61) can be evaluated *analytically* in the BCS and BE limits, by exploiting simplifying features of the calculation. Specifically, the sum over the wave vector in Eqs. (2.60) and (2.64) will be evaluated with the approximation $\Delta_0/|\mu|\ll 1$ that holds in both limits (albeit with $\mu>0$ and $\mu<0$, respectively). The main results of this section are given by Eqs. (2.68) and (2.78) below.

In the weak-coupling (BCS) limit the term $(5/3)(\mathbf{k}^2/2m)(\Delta_{\mathbf{k}}/E_{\mathbf{k}})^2$ within braces in Eq. (2.64) provides the dominant contribution,[42] yielding (in three dimensions)

$$\begin{aligned} b_{\text{BCS}}&=\frac{5}{24m}\sum_{\mathbf{k}} w(\mathbf{k})^2\,\frac{\mathbf{k}^2}{2m}\,\frac{\xi_{\mathbf{k}}^2\Delta_{\mathbf{k}}^2}{E_{\mathbf{k}}^7}\\ &=\Omega\,\frac{5k_Fw(k_F)^2}{48\pi^2\mu\tilde{\Delta}_0^2}\int_{-\tilde{\Delta}_0^{-1}}^{+\infty}dy\,\frac{\tilde{w}(y)^4(y\tilde{\Delta}_0+1)^{3/2}y^2}{[y^2+\tilde{w}(y)^2]^{7/2}}\\ &\cong\Omega\,\frac{5k_Fw(k_F)^2}{48\pi^2\mu\tilde{\Delta}_0^2}\int_{-\infty}^{+\infty}dy\,\frac{y^2}{(y^2+1)^{7/2}}\\ &=\Omega\,\frac{mk_Fw(k_F)^2}{2\pi^2}\,\frac{1}{(3k_F\tilde{\Delta}_0)^2}.\end{aligned} \tag{2.66}$$

In the expression above, we have set $w(k)=w(k_F)\tilde{w}(\tilde{k})$ with $\tilde{k}=k/k_F$ and $\tilde{w}(\tilde{k}=1)=1$, $\xi_k/\mu=\tilde{\xi}_{\tilde{k}}=\tilde{k}^2-1$ (since $\mu=k_F^2/2m$ in the BCS limit), $\Delta_k/\mu=\tilde{w}(\tilde{k})\tilde{\Delta}_0$ with $\tilde{\Delta}_0=w(k_F)\Delta_0/\mu=\Delta_{k_F}/\mu$, and $y=\tilde{\xi}/\tilde{\Delta}_0$. In addition, the last line of Eq. (2.66) has been obtained by exploiting the BCS condition $\tilde{\Delta}_0\ll 1$ as well as the normalization $\tilde{w}(y=0)=\tilde{w}(\tilde{k}=1)=1$. By the same token, the coefficient a given by Eq. (2.60) becomes in the BCS limit (in three dimensions)

$$a_{\rm BCS}=\sum_{\mathbf{k}} w(\mathbf{k})^2 \frac{\Delta_{\mathbf{k}}^2}{2E_{\mathbf{k}}^3}$$
$$=\Omega \frac{mk_F w(k_F)^2}{4\pi^2}\int_{-\tilde{\Delta}_0^{-1}}^{+\infty} dy \frac{\tilde{w}(y)^4(y\tilde{\Delta}_0+1)^{1/2}}{[y^2+\tilde{w}(y)^2]^{3/2}}$$
$$\cong\Omega \frac{mk_F w(k_F)^2}{4\pi^2}\int_{-\infty}^{+\infty} dy \frac{1}{(y^2+1)^{3/2}}$$
$$=\Omega \frac{mk_F w(k_F)^2}{2\pi^2}. \tag{2.67}$$

Entering the results (2.66) and (2.67) into Eq. (2.62), we obtain eventually

$$\xi_{\rm phase}^{\rm BCS}=\sqrt{\frac{b_{\rm BCS}}{a_{\rm BCS}}}=\frac{1}{3k_F\tilde{\Delta}_0}. \tag{2.68}$$

This result has to be compared with the BCS limit for $\xi_{\rm pair}$ [cf. Eqs. (2.8) and (2.9)] obtained previously,[24] namely,

$$\xi_{\rm pair}^{\rm BCS}=\frac{1}{\sqrt{2}k_F\tilde{\Delta}_0}. \tag{2.69}$$

Apart from a numerical factor of order unity due to a different normalization in the respective definitions, $\xi_{\rm phase}$ is thus seen to coincide with $\xi_{\rm pair}$ in the (extreme) BCS limit, as expected. What is less obviously expected, however, is the fact that the ratio $\xi_{\rm phase}/\xi_{\rm pair}$ maintains its BCS value $\sqrt{2}/3 \simeq 0.47$ not only asymptotically (i.e., for $k_F\xi_{\rm pair}\approx 10^3$–$10^4$) but *also* down to $k_F\xi_{\rm pair}\simeq 10$ where bosonization starts to occur, as we shall verify in Sec. IV by calculating the expressions (2.9) and (2.62) numerically.

In the opposite strong-coupling (BE) limit, the term $\xi_{\mathbf{k}}/2$ within braces in Eq. (2.64) provides instead the dominant contribution to the coefficient b,[43] yielding

$$b_{\rm BE}=\frac{1}{8m}\sum_{\mathbf{k}} \frac{w(\mathbf{k})^2}{2\xi_{\mathbf{k}}^{0^2}} \tag{2.70}$$

with $\xi_{\mathbf{k}}^0=\mathbf{k}^2/2m+\epsilon_0/2$ [$-\epsilon_0$ being the (lowest) eigenvalue of the associated eigenvalue problem for two fermions interacting via the potential (2.11)]. By the same token, we obtain for the coefficient a in the BE limit [cf. Eq. (2.60)]:

$$a_{\rm BE}=\sum_{\mathbf{k}} w(\mathbf{k})^2 \frac{\Delta_{\mathbf{k}}^2}{2\xi_{\mathbf{k}}^{0^3}}. \tag{2.71}$$

Upon taking the ratio

$$(\xi_{\rm phase}^{\rm BE})^2=\frac{b_{\rm BE}}{a_{\rm BE}}=\frac{1}{8m}\frac{\Sigma_{\mathbf{k}}[w(\mathbf{k})^2/\xi_{\mathbf{k}}^{0^2}]}{\Sigma_{\mathbf{k}}w(\mathbf{k})^2(\Delta_{\mathbf{k}}^2/\xi_{\mathbf{k}}^{0^3})}\equiv\frac{1}{8\mu_1(2m)}, \tag{2.72}$$

we recognize the quantity μ_1 to be the (positive) shift of the chemical potential (at the lowest significant order in Δ_0/ϵ_0) with respect to the asymptotic value $\mu_0=-\epsilon_0/2<0$. To show this, we resort to the mean-field equation (2.33) in the zero-temperature limit

$$1+V\sum_{\mathbf{k}} \frac{w(\mathbf{k})^2}{2E_{\mathbf{k}}}=0 \tag{2.73}$$

and expand in Δ_0/ϵ_0

$$E_{\mathbf{k}}=\xi_{\mathbf{k}}\left[1+\frac{\Delta_{\mathbf{k}}^2}{2\xi_{\mathbf{k}}^2}+\cdots\right]$$
$$=\left(\frac{\mathbf{k}^2}{2m}-\mu_0-\mu_1+\cdots\right)$$
$$\times\left[1+\frac{1}{2}\frac{\Delta_{\mathbf{k}}^2}{(\mathbf{k}^2/2m-\mu_0-\mu_1+\cdots)^2}+\cdots\right]$$
$$\simeq\left(\frac{\mathbf{k}^2}{2m}-\mu_0\right)+\frac{1}{2}\frac{\Delta_{\mathbf{k}}^2}{(\mathbf{k}^2/2m-\mu_0)}-\mu_1, \tag{2.74}$$

where μ_0 is solution to the bound-state equation

$$1+V\sum_{\mathbf{k}} \frac{w(\mathbf{k})^2}{\mathbf{k}^2/m-2\mu_0}=0 \tag{2.75}$$

that gives $\mu_0=-\epsilon_0/2$. Entering the approximation (2.74) into Eq. (2.73) and expanding further $E_{\mathbf{k}}^{-1}$ at the relevant order yields

$$0=1+V\sum_{\mathbf{k}} \frac{w(\mathbf{k})^2}{2\xi_{\mathbf{k}}^0}\left[1-\frac{1}{2}\frac{\Delta_{\mathbf{k}}^2}{\xi_{\mathbf{k}}^{0^2}}+\frac{\mu_1}{\xi_{\mathbf{k}}^0}+\cdots\right]$$
$$=-\frac{V}{4}\sum_{\mathbf{k}} w(\mathbf{k})^2\frac{\Delta_{\mathbf{k}}^2}{\xi_{\mathbf{k}}^{0^3}}+\mu_1\frac{V}{2}\sum_{\mathbf{k}}\frac{w(\mathbf{k})^2}{\xi_{\mathbf{k}}^{0^2}}+\cdots, \tag{2.76}$$

where use has been made of Eq. (2.75). Solving for the shift μ_1 we obtain eventually

$$\mu_1=\frac{1}{2}\frac{\Sigma_{\mathbf{k}}w(\mathbf{k})^2(\Delta_{\mathbf{k}}^2/\xi_{\mathbf{k}}^{0^3})}{\Sigma_{\mathbf{k}}[w(\mathbf{k})^2/\xi_{\mathbf{k}}^{0^2}]}, \tag{2.77}$$

as anticipated in Eq. (2.72).

The expression (2.72) coincides formally with the (square of the) coherence length associated with a truly bosonic system in the limit of weak boson-boson interaction (or low density), whereby $2\mu_1=v(0)n_B$ [$v(0)$ being the zero-momentum component of the boson-boson interaction and n_B the bosonic density]. In Sec. III we shall obtain an explicit expression for the *residual* boson-boson interaction which results upon bosonization of the original fermionic system, and verify that the product of its zero-momentum component times the bosonic density coincides with the expression (2.77) for $2\mu_1$ (at least, at the one-loop order we are considering in this paper). We regard this result as being a rather compelling check on our one-loop calculation to provide a consistent description of the dilute *interacting* Bose gas, obtained through bosonization of the original fermionic system.

The above results hold for any reasonable choice of the function $w(\mathbf{k})$. With the specific form (2.12) and $\gamma=1/2$, the integrals occurring in Eqs. (2.75) and (2.77) can be performed analytically, yielding (in three dimensions)

$$(k_F \xi^{\mathrm{BE}}_{\mathrm{phase}})^2 = \frac{3\pi}{64}\, \tilde{k}_0 f(c), \tag{2.78}$$

where $\tilde{k}_0 = k_0/k_F$, $c = [|\mu_0|/(k_0^2/2m)]^{1/2}$, and

$$f(c) = \frac{4c}{1+4c} \tag{2.79}$$

is a monotonically increasing function of c bounded between $f(c=0)=0$ and $f(c=\infty)=1$. In particular, $f(c=0.1) \simeq 0.286$. The appropriate value of c [to be inserted in Eq. (2.78)] is then provided by Eq. (2.75), which in the present context reads

$$c = -\frac{V\Omega m k_0}{4\pi} - 1. \tag{2.80}$$

By our assumptions on how the BE limit is achieved (cf. Ref. 43), the value of c is expected to be much smaller than unity, and thus $f(c)$ to be at most of the order 1/3, yielding $\sqrt{\pi \tilde{k}_0}/8$ for the maximum attainable value of $k_F \xi^{\mathrm{BE}}_{\mathrm{phase}}$.

The dependence of $\xi^{\mathrm{BE}}_{\mathrm{phase}}$ on the interaction strength should be contrasted with the value of ξ_{pair} obtained previously in the BE limit (namely, $\xi^{\mathrm{BE}}_{\mathrm{pair}} = r_0$, r_0 being the mean radius of the bound-fermion pair),[24] such that $\xi^{\mathrm{BE}}_{\mathrm{phase}} \gg \xi^{\mathrm{BE}}_{\mathrm{pair}}$ for sensible choices of k_0.[44] Again, this result is consistent with what we had expected in the bosonic limit, where the ''internal'' size r_0 of the bosons represents the smallest length in the problem and is certainly not related with the distance over which the fluctuations of the order parameter correlate.

III. MAPPING ONTO A BOSONIC SYSTEM IN THE STRONG-COUPLING LIMIT

One of the advantages for using the functional-integral approach in the crossover from BCS to BE is that it allows for a *direct* mapping of the original fermionic system in the strong-coupling limit onto an *interacting* bosonic system, at the level of the effective action. The Hubbard-Stratonovich decoupling (2.18) has, in fact, resulted in the effective bosonic action (2.26) for the boson-like complex variables $b(q)$, which resembles a truly bosonic action albeit with an infinite number of couplings. Some caution, however, is in order since the single-particle propagator associated with $b(q)$ lacks the characteristic equal-time step singularity that is expected for a bosonic propagator owing to the bosonic commutator $[b,b^\dagger]=1$. For this reason it will be necessary to reinterpret appropriately the field $b(q)$ in Eq. (2.26), in order to recover a truly bosonic action in the strong-coupling limit.

The purpose of this section is to carry out in detail the mapping onto a bosonic system *at the level of the effective action*, so as to obtain the residual (quartic) interaction among the composite bosons constituted by fermionic pairs.[45] This method will enable us to obtain (at the one-loop order) the phase coherence length for the limiting bosonic system directly in terms of the parameters of the residual interaction, and to compare it with the result obtained for the BE limit in Sec. II. In this way, we will recover the expression for the coherence length of an interacting *dilute* Bose gas,[19] thus establishing a consistency check on the approach of Sec. II.

We begin by considering again the effective bosonic action (2.26) and expand $\mathrm{tr}\ln\mathbf{M}$ therein in powers of $b(q)$ about $b=0$, rather than about the broken-symmetry value $b_0=\beta\Delta_0$ as we did in Eq. (2.28). In addition, we shall not introduce here the loop parameter λ since it is not relevant to the following arguments. We thus split

$$\mathbf{M}(k,k') = \mathbf{M}_S(k,k') + \mathbf{M}_R(k,k'), \tag{3.1}$$

where now

$$\mathbf{M}_S(k,k') = \begin{pmatrix} \epsilon(k) & 0 \\ 0 & -\epsilon(-k) \end{pmatrix} \delta_{k,k'} \tag{3.2}$$

and

$$\mathbf{M}_R(k,k') = w\left(\frac{k+k'}{2}\right) \begin{pmatrix} 0 & \frac{1}{\beta}\, b^*(k-k') \\ \frac{1}{\beta}\, b(k'-k) & 0 \end{pmatrix} \tag{3.3}$$

[cf. Eqs. (2.29)–(2.31)], and obtain

$$S_{\mathrm{eff}} = -\mathrm{tr}\,\ln\mathbf{M}_S - \frac{1}{\beta V}\sum_q |b(q)|^2 + \sum_{n=1}^{\infty} \frac{1}{2n}\, \mathrm{tr} X^{2n} \tag{3.4}$$

with $X = \mathbf{M}_S^{-1}\mathbf{M}_R$:

$$X(k,k') = w\left(\frac{k+k'}{2}\right) \begin{pmatrix} 0 & \dfrac{b^*(k-k')}{\beta\epsilon(k)} \\ -\dfrac{b(k'-k)}{\beta\epsilon(-k)} & 0 \end{pmatrix}. \tag{3.5}$$

We retain first the quadratic terms in Eq. (3.4), that give

$$S^{(2)}_{\mathrm{eff}} = \frac{1}{\beta^2}\sum_q |b(q)|^2 \left[-\frac{\beta}{V} - \sum_k w(k-q/2)^2 \frac{1}{\epsilon(k)\epsilon(q-k)} \right] \tag{3.6}$$

in the place of Eq. (2.36), the expression within brackets coinciding with $A(q)$ given by Eq. (2.37) in the limit $\Delta_0=0$. Keeping the same notation and performing the sum over the fermionic Matsubara frequencies, we obtain

$$A(\mathbf{q},z) = -\frac{\beta}{V} - \frac{\beta}{2}\sum_{\mathbf{k}} w(\mathbf{k}-\mathbf{q}/2)^2 \times \frac{[\tanh(\beta\xi_{\mathbf{k}}/2) + \tanh(\beta\xi_{\mathbf{k}-\mathbf{q}}/2)]}{\xi_{\mathbf{k}} + \xi_{\mathbf{k}-\mathbf{q}} - z}, \tag{3.7}$$

where we have also replaced the bosonic Matsubara frequency $i\omega_\nu$ by the complex frequency z. Viewed as a function of z, $A(\mathbf{q},z)$ has a cut along the real frequency axis for $\mathrm{Re}z \geq -2\mu$ and no other singularity on the (physical) complex plane. In addition, it vanishes for *real* values of z only when $|V|$ is large enough. In this case, we consider the limit $\beta\mu \to -\infty$ and replace Eq. (3.7) by

$$-\frac{V}{\beta} A(\mathbf{q},z) = 1 + V\sum_{\mathbf{k}} \frac{w(\mathbf{k})^2}{\mathbf{k}^2/m + \mathbf{q}^2/4m - 2\mu - z}. \tag{3.8}$$

Let $\omega_{\mathbf{q}}$ be the solution to the equation

$$A(\mathbf{q},\omega_{\mathbf{q}})=0 \tag{3.9}$$

for a given $\mathbf{q}$. Comparison with Eq. (2.75) (where $2\mu_0=-\epsilon_0$ is the bound-state energy) yields

$$\omega_{\mathbf{q}}=\frac{\mathbf{q}^2}{4m}-(2\mu+\epsilon_0). \tag{3.10}$$

It is then clear that the function

$$A'(\mathbf{q},z)\equiv\frac{A(\mathbf{q},z)}{\omega_{\mathbf{q}}-z} \tag{3.11}$$

for a given $\mathbf{q}$ is regular also when $z\to\omega_{\mathbf{q}}$ and nonvanishing over the whole z plane. This remark enables us to rewrite the quadratic action (3.6) in the form

$$S_{\mathrm{eff}}^{(2)}=\frac{1}{\beta^2}\sum_q |b(q)|^2(\omega_{\mathbf{q}}-i\omega_\nu)A'(\mathbf{q},i\omega_\nu), \tag{3.12}$$

which suggests rescaling $b(q)$ by setting

$$b'(q)=\sqrt{A'(\mathbf{q},i\omega_\nu)}\,\frac{b(q)}{\beta}. \tag{3.13}$$

Expressed in terms of the new variables $b'(q)$, the quadratic action (3.12) reduces to that of a noninteracting Bose system with mass $m_B=2m$ and chemical potential $\mu_B=2\mu+\epsilon_0$. Note that the rescaling (3.13) is meaningful insofar as the solution $\omega_{\mathbf{q}}$ to Eq. (3.9) can be found [i.e., for $|V|$ strong enough that the associated two-body problem possesses a bound state, cf. Eq. (2.75)]. In this case, the new field $b'(q)$ acquires the meaning of a truly bosonic field from the $(i\omega_\nu)^{-1}$ decay of its (bare) propagator for large $|\omega_\nu|$ (which, in turn, implies that the correct bosonic commutation rules are recovered for this field).

The rescaling (3.13) obviously affects also the higher ($n>1$) terms of the expansion (3.4), which correspond now to the interacting part of the action for the effective bosonic system with ''free'' action (3.12). In fact, contrary to an ordinary interacting Bose gas for which only the quartic (b^4) interaction exists, bosonization of the original fermionic system has resulted in the infinite set ($b^4,b^6,b^8,\ldots$) of interactions contained in Eq. (3.4). We shall, however, argue that, in the asymptotic limit of a *dilute* Bose gas obtained from bosonization of the original fermionic system when the condition $\epsilon_F/\epsilon_0\ll 1$ is satisfied, it is sufficient to retain only the quartic interaction to obtain all physical quantities of interest. It is thus worth examining first the quartic interaction in some detail.

From Eq. (3.5) we obtain for the term with $n=2$ of Eq. (3.4):

$$S_{\mathrm{eff}}^{(4)}=\frac{1}{4}\,\mathrm{tr}X^4$$
$$=\frac{1}{2}\sum_{q_1\cdots q_4} I_2(q_1\cdots q_4)b'(q_1)^*b'(q_2)^*b'(q_3)b'(q_4) \tag{3.14}$$

with

$$\begin{aligned}I_2(q_1\cdots q_4)&\\ &=\delta_{q_1+q_2,q_3+q_4}\frac{1}{\sqrt{A'(q_1)^*A'(q_2)^*A'(q_3)A'(q_4)}}\\ &\quad\times\sum_k w\left(\frac{2k+q_2}{2}\right)w\left(\frac{2k+q_4}{2}\right)w\left(\frac{2k+2q_2-q_3}{2}\right)\\ &\quad\times w\left(\frac{2k+2q_4-q_1}{2}\right)\\ &\quad\times\frac{1}{\epsilon(-k)\epsilon(k+q_2)\epsilon(k+q_4)\epsilon(-k+q_1-q_4)},\end{aligned} \tag{3.15}$$

where Eq. (3.13) has been used. Comparison with the standard expression for the quartic interaction[20] then yields

$$\frac{1}{\beta\Omega}v_2(q_1\cdots q_4)=I_2(q_1\cdots q_4). \tag{3.16}$$

It is clear from Eq. (3.15) that $v_2(q_1\cdots q_4)$ is, in general, a complicated function of its arguments. What is actually relevant for our purposes, however, is knowing (i) the typical ''strength'' $v_2(0)$ when $q_1=\cdots=q_4=0$ and (ii) the characteristic ''range'' of its Fourier transform in real space. The latter will be examined in Appendix B in the limit $k_0\to\infty$ where calculations get considerably simplified. The strength $v_2(0)$ can be evaluated directly from Eq. (3.15) in the limit $\beta\mu\to-\infty$:

$$\begin{aligned}v_2(0)&=\beta\Omega\,\frac{1}{A'(0)^2}\sum_k w(k)^4\frac{1}{\epsilon(k)^2\epsilon(-k)^2}\\ &=\frac{\beta^2\Omega}{2\pi}\frac{1}{A'(0)^2}\sum_{\mathbf{k}} w(\mathbf{k})^4\int_{-\infty}^{+\infty}d\omega\,\frac{1}{(\omega^2+\xi_{\mathbf{k}}^2)^2}\\ &=\frac{\beta^2\Omega}{4}\frac{1}{A'(0)^2}\sum_{\mathbf{k}}\frac{w(\mathbf{k})^4}{\xi_{\mathbf{k}}^3},\end{aligned} \tag{3.17}$$

where [cf. Eq. (3.11)]

$$A'(0)=\frac{A(0)}{\omega_{\mathbf{q}=0}}=\frac{\beta}{4}\sum_{\mathbf{k}}\frac{w(\mathbf{k})^2}{\xi_{\mathbf{k}}^2} \tag{3.18}$$

at the leading order in the *small* parameter $|\mu_B|/\epsilon_0$. We thus obtain

$$v_2(0)=4\,\frac{(1/\Omega)\Sigma_{\mathbf{k}}[w(k)^4/\xi_{\mathbf{k}}^3]}{((1/\Omega)\Sigma_{\mathbf{k}}[w(k)^2/\xi_{\mathbf{k}}^2])^2}. \tag{3.19}$$

Note that $v_2(0)$ is positive and corresponds to a *repulsive* interaction between the composite bosons. The integrals in Eq. (3.19) can be performed analytically when $\gamma=1/2$ in Eq. (2.12), yielding (in three dimensions)

$$v_2(0)=\frac{32\pi}{2mk_0}\frac{1}{f(c)} \tag{3.20}$$

with $f(c)$ defined by Eq. (2.79). This result enables us to eliminate $f(c)$ in favor of $v_2(0)$ from Eq. (2.78) and rewrite $\xi_{\mathrm{phase}}^{\mathrm{BE}}$ in the form

$$(\xi_{\mathrm{phase}}^{\mathrm{BE}})^2=\frac{1}{4m_B n_B v_2(0)} \tag{3.21}$$

as anticipated in Sec. II, where $n_B=n/2=k_F^3/(6\pi^2)$ is the bosonic density. For completeness, we shall verify below that expression (3.21) coincides with the result obtained for a *dilute* Bose gas with repulsive interaction $v(0)=v_2(0)$.

Before proceeding further and considering the remaining interaction terms (with $n>2$) of Eq. (3.4), it is relevant to discuss the *bosonization condition(s)* which we have exploited in the previous and present sections (namely, $\Delta_0 \ll |\mu|$ with $\mu<0$, see Ref. 43, and $|\mu_B|\ll\epsilon_0$, in the order) to reach the BE limit. (The additional condition $|\mu|\ll k_0^2/2m$ introduced in Ref. 43 is instead related to the specific form of the interaction potential [cf. Eqs. (2.11) and (2.12)], which prohibits probing length scales (such as the bound-state radius r_0) smaller than k_0^{-1} (cf. also Ref. 30). From virial theorem it follows, in fact, that $\epsilon_0\sim r_0^{-2}$, from which $r_0\gg k_0^{-1}$ can be implemented by requiring $k_0^2/2m\gg\epsilon_0=2|\mu|$.) The simplest criterion for dealing with *nonoverlapping* composite bosons is

$$r_0 \ll k_F^{-1} \tag{3.22}$$

(k_F^{-1} identifying the average interparticle distance), from which it follows that $\epsilon_F\ll\epsilon_0$. At zero temperature in the broken-symmetry state, criterion (3.22) is equivalent to $\Delta_0\ll|\mu|$. To show this, we assume that $\Delta_0\ll|\mu|$ and approximate the (mean-field expression for the) density (in dimensions $d<4$) as follows:

$$n=\frac{1}{\Omega}\sum_{\mathbf{k}}\left(1-\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right)\cong\frac{\Delta_0^2}{2\Omega}\sum_{\mathbf{k}}\frac{w(\mathbf{k})^2}{\xi_{\mathbf{k}}^2}\sim\Delta_0^2|\mu|^{d/2-2} \tag{3.23}$$

for sufficiently large k_0. This verifies our assumption consistently, since

$$\frac{\Delta_0^2}{\mu^2}\sim\left(\frac{\epsilon_F}{\epsilon_0}\right)^{d/2}\ll 1. \tag{3.24}$$

At finite temperature, on the other hand, we may use either the Bogolubov result $\mu_B\sim nv_2(0)$ for temperatures lower than the (BE) critical temperature, or the ideal gas value $|\mu_B|\sim n^{2/d}$ for temperatures not too larger than the critical temperature. Taking into account Eq. (3.19), we obtain in the first case

$$|\mu_B|\sim n|\mu|^{1-d/2}\sim|\mu|\left(\frac{\epsilon_F}{|\mu|}\right)^{d/2} \tag{3.25}$$

that gives

$$\frac{|\mu_B|}{\epsilon_0}\sim\left(\frac{\epsilon_F}{\epsilon_0}\right)^{d/2}\ll 1. \tag{3.26}$$

In the second case we obtain instead

$$\frac{|\mu_B|}{\epsilon_0}\sim\frac{n^{2/d}}{\epsilon_0}\sim\frac{\epsilon_F}{\epsilon_0}\ll 1. \tag{3.27}$$

Recall that Eqs. (3.8) and (3.17) have been obtained with the condition $\beta\mu\to-\infty$ (which is equivalent to considering temperatures much smaller than the pair dissociation temperature $\sim\epsilon_0$). Implementing the bosonization criterion $\epsilon_F\ll\epsilon_0$ has required us to introduce, in addition, the BE critical temperature ($\ll\epsilon_0$) and to verify the bosonization criterion in distinct temperature regimes.

There remains to verify that the interaction terms with $n>2$ in Eq. (3.4) can be neglected in comparison with the quartic interaction ($n=2$), whenever the bosonization condition $\epsilon_F\ll\epsilon_0$ is satisfied. To this end, we write in analogy to Eqs. (3.14)–(3.16),

$$\begin{aligned}S_{\mathrm{eff}}^{(2n)}&=\frac{1}{2n}\,\mathrm{tr}X^{2n}\\&=\frac{(\beta\Omega)^{1-n}}{n}\sum_{q_1\cdots q_{2n}}v_n(q_1\cdots q_{2n})b'(q_1)^*\cdots b'(q_n)^*\\&\quad\times b'(q_{n+1})\cdots b'(q_{2n}),\end{aligned} \tag{3.28}$$

and consider only the case $q_1=\cdots=q_{2n}=0$, for which

$$v_n(0)=(-1)^n\left(\frac{\beta\Omega}{A'(0)}\right)^n\frac{1}{\beta\Omega}\sum_k\frac{w(k)^{2n}}{[\epsilon(k)\epsilon(-k)]^n}. \tag{3.29}$$

Note that appropriate powers of $\beta\Omega$ have been introduced in the definition of v_n, in accordance with the standard requirement on the Fourier transform of a generic interaction potential in perturbation theory (no care has, however, been paid

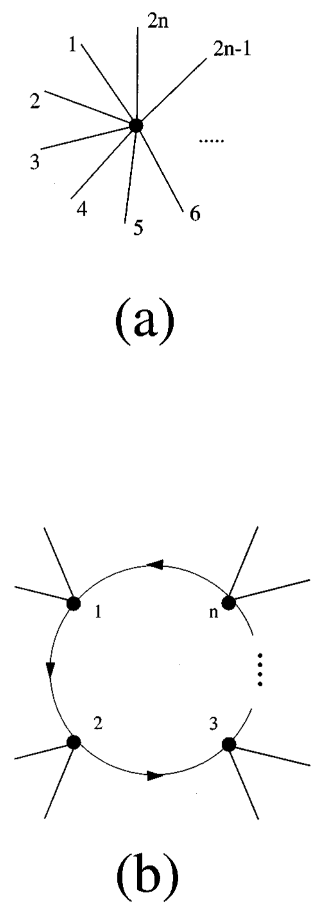

FIG. 4. Comparison of (a) the n-body interaction $v_n(0)$ with (b) the ‘‘effective’’ n-boson interaction L_n constructed from $v_2(0)$.

to symmetrizing v_n). In this way, the expression (3.29) is finite in the limits $\Omega\to\infty$ and/or $\beta\to\infty$ [cf. Eq. (3.18)]. To get an *estimate* of $v_n(0)$, we consider the case $w(k)=1$ and perform the frequency sum in the limit $\beta\mu\to-\infty$:

$$\begin{aligned} v_n(0) &\sim \left(\frac{1}{\Omega}\sum_{\mathbf{k}}\frac{1}{\xi_{\mathbf{k}}^2}\right)^{-n}\frac{1}{\Omega}\sum_{\mathbf{k}}\int_{-\infty}^{+\infty}d\omega\,\frac{1}{(\omega^2+\xi_{\mathbf{k}}^2)^n} \\ &\sim \left(\frac{1}{\Omega}\sum_{\mathbf{k}}\frac{1}{\xi_{\mathbf{k}}^2}\right)^{-n}\frac{1}{\Omega}\sum_{\mathbf{k}}\xi_{\mathbf{k}}^{1-2n} \\ &\sim (|\mu|^{d/2-2})^{-n}|\mu|^{d/2-2n+1} \\ &= |\mu|^{(d/2)(1-n)+1} \end{aligned} \tag{3.30}$$

in d dimensions.[46]

Neglecting the interactions $v_n(0)$ with $n>2$ with respect to $v_2(0)$ relies on the following argument. Consider the ''effective'' n-boson interaction L_n depicted in Fig. 4, which is assembled from $n>2$ (bare) bosonic propagators arranged in a loop and n interactions $v_2(0)$:

$$L_n=(v_2(0))^n\,\frac{1}{\beta\Omega}\sum_q\frac{1}{(i\omega_\nu-\omega_{\mathbf{q}})^n} \tag{3.31}$$

with $\omega_{\mathbf{q}}$ given by Eq. (3.10) and $\mu_B<0$. For temperatures not too larger than the (BE) critical temperature [for which Eq. (3.27) holds], it can be readily shown that

$$\frac{1}{\beta\Omega}\sum_q\frac{1}{(i\omega_\nu-\omega_{\mathbf{q}})^n}\sim\beta^{n-d/2-1}\sim|\mu_B|^{d/2-n+1} \tag{3.32}$$

with a *finite* constant of proportionality. [Note that the result (3.32) could have been guessed directly from dimensional analysis.] Comparison with Eq. (3.30) yields eventually

$$\frac{v_n(0)}{L_n}\sim\frac{|\mu|^{d/2(1-n)+1}}{(|\mu|^{1-d/2})^n|\mu_B|^{d/2-n+1}}\sim\left|\frac{\mu_B}{\epsilon_0}\right|^{n-d/2-1}\ll 1 \tag{3.33}$$

for $n\geqslant 3$ and $d<4$, whenever the bosonization criterion $\epsilon_F\ll\epsilon_0$ is satisfied. In this case it is clear that all physical quantities can be obtained by retaining the quartic interaction (v_2) only. Below the critical temperature, on the other hand, the Bogolubov propagator has to be used in Eq. (3.31) in the place of the bare bosonic propagator. This replacement leads to (infrared) divergent integrals, which have to be handled by a suitable renormalization procedure. Although one might argue that an *infinite* constant of proportionality on the right-hand side of Eq. (3.32) would make the condition $v_n(0)/L_n\ll 1$ be satisfied *a fortiori*, consideration of the renormalization procedure is beyond the purposes of the present paper. This gives a clear warning that a complete description of the crossover from BCS to BE unavoidably requires one to face the peculiar problems arising in the bosonic limit.

In conclusion, we have shown that the action (3.4) can be reduced in the bosonization limit to the simpler form:

$$\begin{aligned} S_{\text{eff}} &= \sum_q |b'(q)|^2(\omega_{\mathbf{q}}-i\omega_\nu)+\frac{1}{2\beta\Omega} \\ &\quad\times\sum_{q_1\cdots q_4} v_2(q_1\cdots q_4)b'(q_1)^*b'(q_2)^*b'(q_3)b'(q_4) \end{aligned} \tag{3.34}$$

(apart from the constant term $-\mathrm{tr}\ln\mathbf{M}_S$), where $v_2(q_1\cdots q_4)$ depends on its arguments in a complicated way [cf. Eq. (3.15)]. Nonetheless, for many purposes it should be possible to neglect ''retardation'' effects and replace $v_2(q_1\cdots q_4)$ with $v_2(0)$ given by Eq. (3.19). In that case, the *mapping* of the original fermionic system *onto a truly bosonic system* is fully established.[47]

There remains to recall how the expression (3.21) for ξ_{phase} at zero temperature in the bosonic limit can be obtained *directly* from the bosonic action (3.34) with a constant $v_2(0)$. To this end, we set, as usual, $b'(q)=\sqrt{\beta\Omega}\,\alpha\,\delta_{q,0}+\underset{\sim}{b}'(q)$ and expand (3.34) up to quadratic order in $\underset{\sim}{b}'(q)$:

$$\begin{aligned} S_{\text{eff}} &\cong \beta\Omega|\alpha|^2\left(\frac{1}{2}v_2(0)|\alpha|^2-\mu_B\right)+\frac{1}{2}\sum_q(\underset{\sim}{b}'^*(q),\underset{\sim}{b}'(-q))\begin{pmatrix}\frac{\mathbf{q}^2}{2m_B}-i\omega_\nu+|\alpha|^2v_2(0), & \alpha^2v_2(0) \\ \alpha^{*2}v_2(0), & \frac{\mathbf{q}^2}{2m_B}+i\omega_\nu+|\alpha|^2v_2(0)\end{pmatrix} \\ &\quad\times\begin{pmatrix}\underset{\sim}{b}'(q) \\ \underset{\sim}{b}'^*(-q)\end{pmatrix}, \end{aligned} \tag{3.35}$$

where the Bogolubov self-consistency condition $\mu_B=v_2(0)|\alpha|^2$ has been used. The single-particle bosonic propagators can then be readily obtained by inverting the Gaussian matrix in Eq. (3.35), yielding (in matrix form):

$$\left\langle\begin{pmatrix}\underset{\sim}{b}'(q) \\ \underset{\sim}{b}'^*(-q)\end{pmatrix}(\underset{\sim}{b}'^*(q),\underset{\sim}{b}'(-q))\right\rangle_{S_{\text{eff}}}=\frac{1}{\omega_\nu^2+E_{\mathbf{q}}^2}\begin{pmatrix}\frac{\mathbf{q}^2}{2m_B}+i\omega_\nu+|\alpha|^2v_2(0) & -\alpha^2v_2(0) \\ -\alpha^{*2}v_2(0) & \frac{\mathbf{q}^2}{2m_B}-i\omega_\nu+|\alpha|^2v_2(0)\end{pmatrix} \tag{3.36}$$

with the Bogolubov quasiparticle dispersion $E_{\mathbf{q}} = \sqrt{(\mathbf{q}^2/2m_B)^2 + 2|\alpha|^2 v_2(0)\mathbf{q}^2/2m_B}$. We are actually interested in the longitudinal and transverse correlation functions, defined as [cf. Eqs. (2.4)]

$$G^{\parallel}(q) = \langle \underset{\sim}{b}'_{\parallel}(q)\underset{\sim}{b}'_{\parallel}(-q)\rangle, \tag{3.37a}$$

$$G^{\perp}(q) = \langle \underset{\sim}{b}'_{\perp}(q)\underset{\sim}{b}'_{\perp}(-q)\rangle, \tag{3.37b}$$

where [cf. Eqs. (2.2)]

$$\underset{\sim}{b}'_{\parallel}(q) = \frac{1}{2|\alpha|}\,[\alpha^* \underset{\sim}{b}'(q) + \alpha \underset{\sim}{b}'^*(-q)], \tag{3.38a}$$

$$\underset{\sim}{b}'_{\perp}(q) = \frac{1}{2i|\alpha|}\,[\alpha^* \underset{\sim}{b}'(q) - \alpha \underset{\sim}{b}'^*(-q)]. \tag{3.38b}$$

In terms of the propagators (3.36) we obtain (for real α)

$$G^{\parallel}(q) = \frac{\mathbf{q}^2/(4m_B)}{\omega_\nu^2 + E_{\mathbf{q}}^2}, \tag{3.39a}$$

$$G^{\perp}(q) = \frac{\mathbf{q}^2/(4m_B) + \alpha^2 v_2(0)}{\omega_\nu^2 + E_{\mathbf{q}}^2}. \tag{3.39b}$$

In particular, the zero-frequency correlation functions read

$$G^{\parallel}(\mathbf{q},\omega_\nu = 0) = \frac{1}{4m_B}\frac{\mathbf{q}^2}{E_{\mathbf{q}}^2} \cong \frac{1}{4m_B v_S^2}\frac{1}{1 + \mathbf{q}^2\xi_{\text{phase}}^2}, \tag{3.40a}$$

$$G^{\perp}(\mathbf{q},\omega_\nu = 0) = \frac{\mathbf{q}^2/(4m_B) + \alpha^2 v_2(0)}{E_{\mathbf{q}}^2} \cong \frac{\alpha^2 v_2(0)}{v_S^2 \mathbf{q}^2} \tag{3.40b}$$

with the approximate expressions on the right-hand side holding in the small-$\mathbf{q}$ limit, whereby

$$E_{\mathbf{q}} \cong v_S|\mathbf{q}|\sqrt{1 + \mathbf{q}^2\xi_{\text{phase}}^2}. \tag{3.41}$$

Here $v_S = \sqrt{\alpha^2 v_2(0)/m_B}$ is the Bogolubov sound velocity and $\xi_{\text{phase}} = [4m_B\alpha^2 v_2(0)]^{-1/2}$ is the desired coherence length for longitudinal correlations. Recalling further that the ''condensate'' density α^2 coincides with the particle density n_B in the Bogolubov approximation, expression (3.21) (obtained in the bosonization limit) is eventually recovered. This completes our mapping. In the next section we will show numerically how the crossover for ξ_{phase} progresses from the BCS value (2.68) to the BE value (3.21).

IV. NUMERICAL RESULTS AND DISCUSSION

In Sec. II we have identified the coherence length ξ_{phase}, associated with the phase-phase correlation function of a superconducting fermionic system with attractive interaction (2.11), as given by Eq. (2.62) together with Eqs. (2.60) and (2.64). In that section we have also evaluated analytically the asymptotic expressions of ξ_{phase} in the extreme (weak- and strong-coupling) limits. There remains to obtain the behavior of ξ_{phase} in the intermediate-coupling regime, which is especially relevant for the crossover between the two limits. In this regime Eqs. (2.60) and (2.64) have to be evaluated numerically.

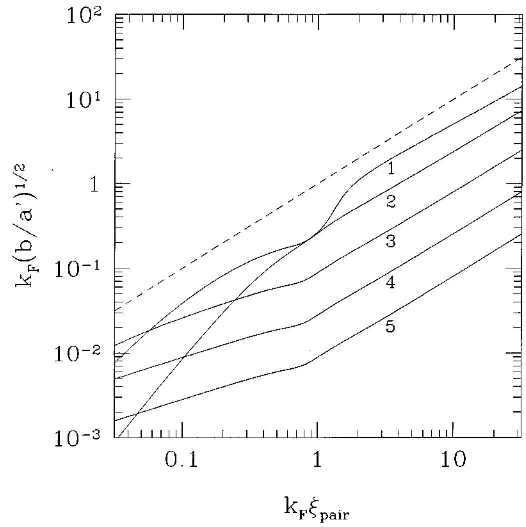

FIG. 5. Range $(b/a')^{1/2}$ of the function (4.1) (in units of k_F^{-1}) calculated in three dimensions vs $k_F\xi_{\text{pair}}$ for $\gamma=1/2$ and several values of k_0/k_F (defined by 10^{N-1} with $N=1,2,\ldots,5$). The line $k_F\xi_{\text{pair}}$ is also shown for comparison (dashed line).

To this end, the mean-field parameters μ and Δ_0 need to be obtained first. In Appendix C μ and Δ_0 are conveniently expressed in terms of the variable $k_F\xi_{\text{pair}}$ in the special case $k_0=\infty$. A similar scheme can be used for *finite* values of k_0, for which μ and Δ_0 depend *also* on the parameter k_0/k_F. The behavior of μ vs $k_F\xi_{\text{pair}}$ for a wide range of values of k_0/k_F has been already given in Ref. 24 and is reported for the sake of comparison in Appendix D for $k_0=\infty$.

The only mean-field quantity to be discussed here is the expression (2.54) for the longitudinal correlation function. Since the quantity within brackets (with the choice of the minus sign) in Eq. (2.54) coincides with twice the square of the *coherence factor* $(u_{\mathbf{k}}u_{\mathbf{k-q}} - v_{\mathbf{k}}v_{\mathbf{k-q}})$ entering expression (2.58), it is evident by inspection that

$$F_E^{\parallel}(\mathbf{R}) = \frac{V^2}{8}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{R}}\left(-\frac{1}{V} - f(\mathbf{q})\right) \approx \frac{V^2}{8}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{R}}\frac{a'^2}{a' + b\mathbf{q}^2} \tag{4.1}$$

with $f(\mathbf{q})$ given by Eq. (2.61) and

$$a' = \sum_{\mathbf{k}} w(\mathbf{k})^2 \frac{\xi_{\mathbf{k}}^2}{2E_{\mathbf{k}}^3}. \tag{4.2}$$

This identifies the characteristic ''range'' of the function $F_E^{\parallel}(\mathbf{R})$ with $(b/a')^{1/2}$ [cf. Eq. (2.62) and Ref. 41], which is reported vs $k_F\xi_{\text{pair}}$ in Fig. 5 for the choice $\gamma=1/2$ in Eq. (2.12). As anticipated in Sec. II, $F_E^{\parallel}(\mathbf{R})$ can be considered to be a ''short-range'' function of $\mathbf{R}$, since its range never exceeds ξ_{pair} in the BE limit and vanishes when $k_0\to\infty$. For this reason, the coherence length ξ_{phase} cannot be identified at the mean-field level.

The ''long-range'' coherence length of interest can be identified instead by the one-loop calculation of Sec. II according to Eq. (2.62). In Fig. 6 $k_F\xi_{\text{phase}}$ is shown vs $k_F\xi_{\text{pair}}$ for $\gamma=1/2$ and several values of k_0/k_F (full lines). Also shown in the figure are (a) the asymptotic curve (thick line) corresponding to the $k_0=\infty$ calculation of Appendix C; (b)

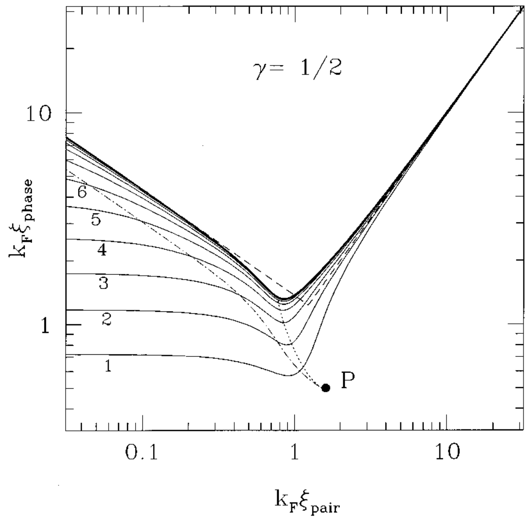

FIG. 6. $k_F\xi_{\rm phase}$ vs $k_F\xi_{\rm pair}$ for $\gamma=1/2$ and several values of k_0/k_F [defined by $10^{N/3}$ with $N=0,1,\ldots,10$, such that the ''reduced'' density is $(k_0/k_F)^{-3}=10^{-N}$]. Values of N label different full curves. Additional conventions are specified in the text.

the boundary (dashed-dotted line) of the ''physical'' region identified by $c=1/4$ with c defined after Eq. (2.78); (c) the curve corresponding to $\mu=0$ (dotted line) where any remnant of the Fermi surface has definitely disappeared; (d) the extrapolation for $k_0=\infty$ of the analytic BCS and BE results obtained in Sec. II C (dashed line). Note that the values of $\xi_{\rm phase}$ have been uniformly multiplied by the factor $3/\sqrt{2}$ to make $\xi_{\rm phase}$ coinciding with $\xi_{\rm pair}$ in the BCS limit, taking into account their different definitions.

Note also the following features from Fig. 6.

(i) $\xi_{\rm phase}$ coincides with $\xi_{\rm pair}$ (*irrespective* of k_0) not only asymptotically in the BCS limit but *also* down to $k_F\xi_{\rm pair}\simeq 10$, where $|\xi_{\rm phase}-\xi_{\rm pair}|/\xi_{\rm pair}<0.03$ for the values of k_0 reported in the figure.

(ii) For $k_F\xi_{\rm pair}\lesssim 10$ there appears a dependence on k_0, which becomes quite pronounced in the BE limit.

(iii) For given k_0, the minimum value of $\xi_{\rm phase}$ occurs (approximately) at $\mu=0$ (dotted line).

(iv) The ''physical'' boundary (dashed-dotted line) and the asymptotic $k_0=\infty$ curve (thick line) delimit a rather narrow strip for $\xi_{\rm phase}$.

(v) There exists an accumulation point (denoted by P in the figure) to which the results for $\mu=0$ converge when $k_0\to 0$. P belongs also to the ''physical'' boundary.

(vi) The extrapolation for $k_0\to\infty$ of the analytic BCS and BE results (dashed line) coincides with the asymptotic $k_0=\infty$ curve (thick line) *except for* a rather narrow region about $k_F\xi_{\rm pair}\simeq 1$ (or $\mu=0$). The region where the two curves depart from each other coincides approximately with the ''intermediate'' region identified in three dimensions from Fig. 11 in Appendix D.

Figure 6 summarizes the main results of this section. For completeness, we also report in Fig. 7 the behavior of $k_F\xi_{\rm phase}$ vs $k_F\xi_{\rm pair}$ using two different values ($\gamma=0.4$, 0.6) for the exponent of Eq. (2.12). Note that the conclusions (i)–(vi) drawn above for $\gamma=1/2$ remain valid, the main difference among results with different values of γ residing in the way they depart from the ''physical'' boundary. In this sense, the value $\gamma=1/2$ (considered by NSR) appears to be special.

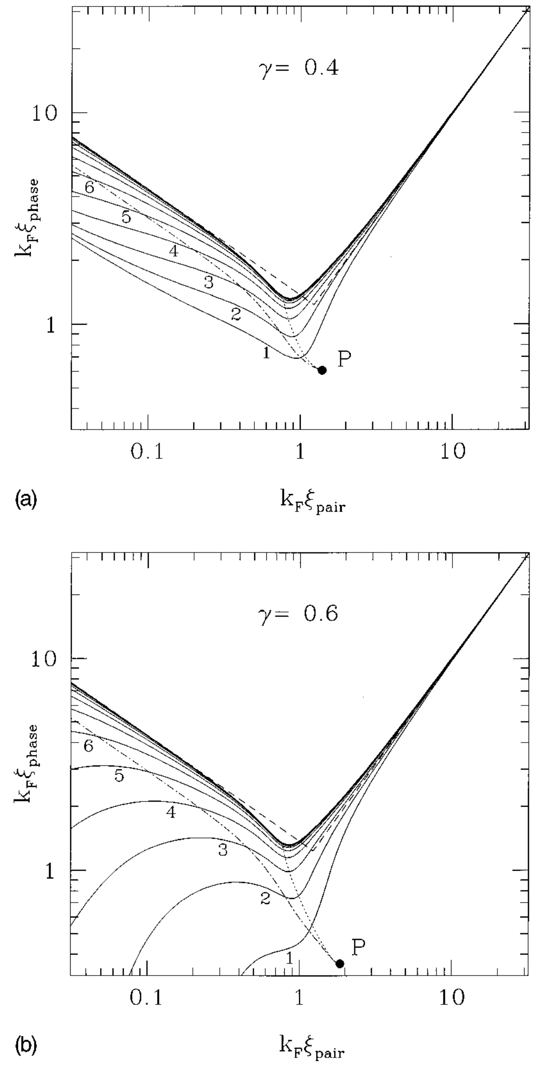

FIG. 7. $k_F\xi_{\rm phase}$ vs $k_F\xi_{\rm pair}$ for (a) $\gamma=0.4$ and (b) $\gamma=0.6$. Conventions are as in Fig. 6.

All results reported above hold specifically in three dimensions. Results of $k_F\xi_{\rm phase}$ vs $k_F\xi_{\rm pair}$ for smaller values of the dimensionality ($2\leqslant d\leqslant 3$) can be obtained by the method of Appendix C in the case $k_0=\infty$ and are reported in Fig. 8. The results for $d=2$, however, have to be interpreted with caution since fluctuation effects (over and above those considered in the present paper) are especially effective in low dimensionality. Note, finally, from Fig. 8 that the value $k_F\xi_{\rm pair}=10$ is still special, since it is (approximately) where the results obtained (with given value of k_0) for different dimensionalities begin to deviate from each other.

V. CONCLUDING REMARKS

In this paper we have described the zero-temperature behavior of the length $\xi_{\rm phase}$ associated with the fluctuations of the superconducting order parameter, following its crossover from BCS to BE limits. Since the breaking of the gauge symmetry is the phenomenon underlying both superconductivity and superfluidity,[48] determining how $\xi_{\rm phase}$ crosses over

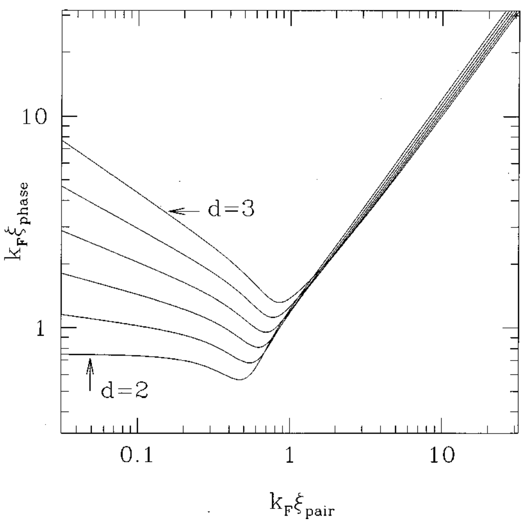

FIG. 8. $k_F \xi_{\rm phase}$ vs $k_F \xi_{\rm pair}$ for $k_0 = \infty$ and intermediate values of the dimensionality d (in steps of 0.2).

between the two limits is of definite relevance.

$\xi_{\rm phase}$ has been contrasted with the particle-correlation length $\xi_{\rm pair}$, which serves to identify the BCS and BE limits as well as to specify the dynamical evolution in between. We have found that $\xi_{\rm phase}$ coincides with $\xi_{\rm pair}$ in the BCS limit and that $\xi_{\rm phase} \gg \xi_{\rm pair}$ in the BE limit, with an interesting behavior in between.

In our calculation we have identified $\xi_{\rm phase}$ and $\xi_{\rm pair}$ using definitions which are valid, in both cases, at the respective significant orders. Specifically, $\xi_{\rm pair}$ has been obtained at the mean-field level and $\xi_{\rm phase}$ at the one-loop order. Some final comments on this procedure, which deals with $\xi_{\rm phase}$ and $\xi_{\rm pair}$ on a different footing, are in order.

$\xi_{\rm pair}$ can be extracted from the fermionic pair-correlation function $g(\mathbf{r})$ defined by Eq. (2.7). Knowledge of $\xi_{\rm pair}$, in turn, exhausts all relevant information contained in $g(\mathbf{r})$ whenever the underlying dynamical problem possesses a *single* characteristic length. We have verified that this is the case when $g(\mathbf{r})$ is calculated at the mean-field level [cf. Eq. (2.8)], whereby $\xi_{\rm pair}$ reduces to the bound-state radius r_0 in the BE limit. This property, however, might not remain true when fluctuations are included, i.e., by calculating $g(\mathbf{r})$ at the one-loop order via Eq. (2.7). In this case, in fact, a second characteristic length (namely, $\xi_{\rm phase}$) is expected to appear in $g(\mathbf{r})$. Unfortunately, it is not possible to verify explicitly how $\xi_{\rm phase}$ enters $g(\mathbf{r})$ at the one-loop level by the method of Sec. II for evaluating the fermionic-correlation functions. It is, in fact, known from the work of Ref. 5 that determining the *density* response function [of which $g(\mathbf{r})$ is a particular case] requires one to include also the coupling between the density and phase-amplitude fluctuations, thus enlarging the Gaussian matrix of Eq. (2.36). Taking into account this coupling exceeds the purposes of the present paper. In any event, it should be sufficient for our purposes to identify $\xi_{\rm pair}$ at the mean-field level, since on physical ground no appreciable change is expected for the smallest length scale in the problem when including fluctuations.

We have further argued that $\xi_{\rm phase}$, on the other hand, cannot be defined at the mean-field level, requiring one to consider explicitly the (one-loop) fluctuation corrections to the longitudinal correlator. It is known, however, that the longitudinal correlator is strongly coupled to the (singular) transverse correlator already at the *next* order of the loop expansion.[49] As a consequence, the longitudinal correlator itself develops singularities for small momenta.[50] Nevertheless, it can be argued that (at least in the bosonic limit) a characteristic length can still be extracted from the longitudinal correlator, this length being identified with our $\xi_{\rm phase}$.[34] For these reasons, our results for $\xi_{\rm phase}$ vs $\xi_{\rm pair}$ are expected to be essentially correct, both lengths being stable against the inclusion of higher-order fluctuations.

Following our approach to the bosonization problem stated in the Introduction, a deeper understanding of the BE limit should greatly help describing also the crossover problem. As mentioned above, proper treatment of the interacting-boson problem requires special care, owing to the occurrence of infrared singularities that strongly affect the calculation of physical quantities. The study of the large-scale behavior of the bosonic propagator, together with the occurrence of intrinsic infrared singularities, then naturally leads us to consider a renormalization-group approach for handling these singularities. Work along these lines is still required for a full understanding of the interacting-boson problem. Such an approach, besides being useful for treating the original bosonization problem in the crossover region, has also renewed interest on its own after the recent discovery of a Bose-condensed system.[51]

ACKNOWLEDGMENTS

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APPENDIX A: BROKEN-SYMMETRY PARAMETER AT THE ONE-LOOP ORDER

In this Appendix we prove the identity (2.44), relating the broken-symmetry parameter Δ [cf. Eq. (2.43)] of the original fermionic system to the average of the $q=0$ component of the bosonic-like variables $b(q)$ introduced via the transformation (2.18). We shall also obtain an explicit expression for the shift Δ_1 of Δ at the one-loop order. Although the explicit value of Δ_1 is irrelevant for the calculation of the phase coherence length of Sec. II, Δ_1 enters in general the expressions of thermodynamic quantities and correlation functions other than (2.4), for which omitting Δ_1 might lead to inconsistencies.[13,22]

The identity (2.44) is proved by adding to the original fermionic action (2.14) the following bosonic-like source term:

$$\delta S = -J_0 \sum_k w(k) \overline{c_\uparrow}(k) \overline{c_\downarrow}(-k) - J_0^* \sum_k w(k) c_\downarrow(-k) c_\uparrow(k) \tag{A1}$$

[with $k = (\mathbf{k}, \omega_s)$]. In this way, one obtains from the resulting generating functional analogous to (2.13):

$$\left\langle \sum_k w(k) c_\downarrow(-k) c_\uparrow(k) \right\rangle_S = \left. \frac{\delta Z[\overline{\eta}, \eta; J_0^*, J_0]}{\delta J_0^*} \right|_{\substack{J_0 = J_0^* = 0 \\ \eta = \overline{\eta} = 0}} \tag{A2}$$

where the average on the left-hand side is evaluated with the action (2.14). On the other hand, when introducing the Hubbard-Stratonovich transformation (2.18) the additional term (A1) can be reabsorbed by shifting the integration variable $b(q)$ with $q=0$, i.e., by setting $b'(q=0) = b(q=0) + \beta J_0$. In this way, one readily obtains

$$-\frac{1}{V} \langle b(q=0) \rangle_{S_{\text{eff}}} = \left. \frac{\delta Z[\overline{\eta}, \eta; J_0^*, J_0]}{\delta J_0^*} \right|_{\substack{J_0 = J_0^* = 0 \\ \eta = \overline{\eta} = 0}}, \tag{A3}$$

where now the average on the left-hand side is evaluated with the action (2.26). Comparison of (A3) with (A2) yields eventually the result (2.44).

The calculation of the (one-loop) shift Δ_1 is thus equivalent to that of the (one-loop) shift $b_1 = \beta \Delta_1$ of $\langle b(q=0) \rangle_{S_{\text{eff}}}$. According to a general procedure of functional integrals (cf., e.g., Appendix C of Ref. 39), this shift is given by

$$b_1 = -e^{i\varphi_0} \frac{\partial F_1 / \partial |b_0|}{\partial^2 F_0 / \partial |b_0|^2}, \tag{A4}$$

where φ_0 is the phase of the source J_0 in Eq. (A1), F_0 and F_1 are given by Eqs. (2.35) and (2.41), respectively, and $b_0 = \beta \Delta_0$.

Alternatively, Δ_1 can be evaluated in terms of the diagrammatic structure for the original fermionic system via the definition (2.44). This procedure has been used in deriving Eq. (2.55), where the first term on the right-hand side was obtained from the first term on the right-hand side of Eq. (2.50). It is interesting to show explicitly the equivalence of the two procedures at the one-loop order. Besides providing a nontrivial consistency check on our one-loop calculation, the following results may also serve, e.g., to obtain the one-loop correction to the chemical potential over and above its mean-field value.

We begin by writing the single-particle fermionic Green's functions in the form

$$\langle c_\uparrow(k) \overline{c_\uparrow}(k) \rangle_S = -\langle \mathbf{M}^{-1}(k,k)_{11} \rangle_{S_{\text{eff}}},$$

$$\langle c_\downarrow(k) \overline{c_\downarrow}(k) \rangle_S = \langle \mathbf{M}^{-1}(-k,-k)_{22} \rangle_{S_{\text{eff}}}, \tag{A5}$$

$$\langle c_\uparrow(k) c_\downarrow(-k) \rangle_S = -\langle \mathbf{M}^{-1}(k,k)_{21} \rangle_{S_{\text{eff}}},$$

where $\mathbf{M}^{-1}$ is the inverse of the matrix (2.23). We can then express the particle density in the form

$$n = \frac{1}{\beta \Omega} \sum_{k,\sigma} e^{i\omega_s \delta} \langle \overline{c_\sigma}(k) c_\sigma(k) \rangle_S$$

$$= \frac{1}{\beta \Omega} \sum_k e^{i\omega_s \delta} (\langle \mathbf{M}^{-1}(k,k)_{11} \rangle_{S_{\text{eff}}} - \langle \mathbf{M}^{-1}(-k,-k)_{22} \rangle_{S_{\text{eff}}}) \tag{A6}$$

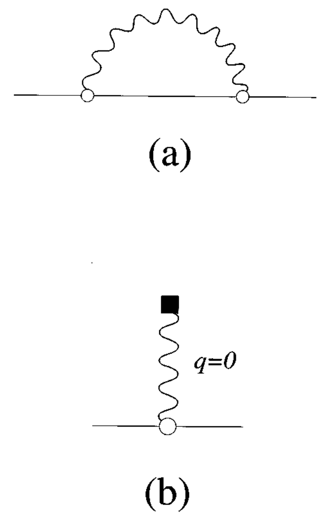

FIG. 9. (a) Graphical representation of a typical term of Eq. (A9) (conventions are as in Fig. 2); (b) zero-momentum insertion of Eq. (A10).

($\delta = 0^+$), and the order parameter Δ in the form

$$\Delta = \frac{V}{\beta} \sum_k w(k) \langle c_\uparrow(k) c_\downarrow(-k) \rangle_S$$

$$= -\frac{V}{\beta} \sum_k w(k) \langle \mathbf{M}^{-1}(k,k)_{21} \rangle_{S_{\text{eff}}}. \tag{A7}$$

Approximations are introduced at this point in the usual way, by (i) replacing $S_{\text{eff}} \to S_{\text{eff}}/\lambda$, (ii) implementing the λ expansion via Eqs. (2.28) and (2.29), and (iii) expanding the resulting expressions in powers of λ. To the first significant order in λ beyond mean field we obtain

$$\langle \mathbf{M}_\lambda^{-1}(k,k')_{ii'} \rangle_{S_{\text{eff}}/\lambda}$$

$$\cong \delta_{k,k'} \mathbf{M}_0^{-1}(k)_{ii'} + \lambda \langle [\mathbf{M}_0^{-1} \mathbf{M}_1 \mathbf{M}_0^{-1} \mathbf{M}_1 \mathbf{M}_0^{-1}]_{kk'}^{ii'} \rangle_{S_{\text{eff}}^{(2)}}$$

$$-\frac{\lambda}{3} \langle [\mathbf{M}_0^{-1} \mathbf{M}_1 \mathbf{M}_0^{-1}]_{kk'}^{ii'}\ \mathrm{tr}(\mathbf{M}_0^{-1} \mathbf{M}_1)^3 \rangle_{S_{\text{eff}}^{(2)}}, \tag{A8}$$

where $S_{\text{eff}}^{(2)}$ is the (λ independent) quadratic action (2.36) and the trace is performed over the indices k and i. In deriving Eq. (A8) we have used the expansion (2.49) for $\mathbf{M}_\lambda^{-1}$ and included consistently the cubic ($n=3$) term in Eq. (2.32). We shall verify that the first term of order λ on the right-hand side of Eq. (A8) represents a nontrivial self-energy correction to the bare propagator $\mathbf{M}_0^{-1}$, while the second term of order λ results by shifting the mean-field parameter $\Delta_0 \to \Delta_0 + \Delta_1$ in $\mathbf{M}_0^{-1}$.

There remains to evaluate the contractions entering Eq. (A8). To this end, it is convenient to supplement the Gaussian action (2.36) by the source term $V^{-1}[J_0 b'(q=0)^* + J_0^* b'(q=0)]$ [which is equivalent to Eq. (A1)], in order to avoid spurious divergencies due to the presence of the Goldstone mode at $q=0$, allowing J_0 to vanish at the end of the calculation. We eventually obtain

$$\langle[\mathbf{M}_0^{-1}\mathbf{M}_1\mathbf{M}_0^{-1}\mathbf{M}_1\mathbf{M}_0^{-1}]_{kk'}^{ii'}\rangle_{S_{\mathrm{eff}}^{(2)}}=\delta_{k,k'}\Bigg\{\mathbf{M}_0^{-1}(k)_{i1}\Bigg[\sum_q w(k+q/2)^2\mathbf{M}_0^{-1}(k+q)_{21}\langle \underset{\sim}{b}^*(q)\underset{\sim}{b}^*(-q)\rangle_{S_{\mathrm{eff}}^{(2)}}\Bigg]\mathbf{M}_0^{-1}(k)_{2i'}$$
$$+\mathbf{M}_0^{-1}(k)_{i1}\Bigg[\sum_q w(k+q/2)^2\mathbf{M}_0^{-1}(k+q)_{22}\langle \underset{\sim}{b}(-q)\underset{\sim}{b}^*(-q)\rangle_{S_{\mathrm{eff}}^{(2)}}\Bigg]\mathbf{M}_0^{-1}(k)_{1i'}$$
$$+\mathbf{M}_0^{-1}(k)_{i2}\Bigg[\sum_q w(k+q/2)^2\mathbf{M}_0^{-1}(k+q)_{11}\langle \underset{\sim}{b}(q)\underset{\sim}{b}^*(q)\rangle_{S_{\mathrm{eff}}^{(2)}}\Bigg]\mathbf{M}_0^{-1}(k)_{2i'}$$
$$+\mathbf{M}_0^{-1}(k)_{i2}\Bigg[\sum_q w(k+q/2)^2\mathbf{M}_0^{-1}(k+q)_{12}\langle \underset{\sim}{b}(q)\underset{\sim}{b}(-q)\rangle_{S_{\mathrm{eff}}^{(2)}}\Bigg]\mathbf{M}_0^{-1}(k)_{1i'}\Bigg\} \tag{A9}$$

depicted schematically in Fig. 9(a), and

$$-\frac{1}{3}\langle[\mathbf{M}_0^{-1}\mathbf{M}_1\mathbf{M}_0^{-1}]_{kk'}^{ii'}\ \mathrm{tr}(\mathbf{M}_0^{-1}\mathbf{M}_1)^3\rangle_{S_{\mathrm{eff}}^{(2)}}$$
$$=\delta_{k,k'}\left\{\frac{\partial\mathbf{M}_0^{-1}(k)_{ii'}}{\partial b_0^*}\,b_1^*+\frac{\partial\mathbf{M}_0^{-1}(k)_{ii'}}{\partial b_0}\,b_1\right\} \tag{A10}$$

depicted in Fig. 9(b), with b_1 given by Eq. (A4).

In particular, when $i=2$ and $i'=1$, entering (A9) and (A10) into (A8) with $\lambda=1$ and the resulting expression into (A7) yields

$$\Delta-\Delta_0=-\frac{V}{\beta}\,h+\frac{V}{\beta}\left[B(q=0)\,\frac{b_1^*}{\beta}+\left(A(q=0)+\frac{\beta}{V}\right)\frac{b_1}{\beta}\right]$$
$$=\frac{b_1}{\beta}-\frac{V}{\beta}\left[h-[|B(q=0)|+A(q=0)]\,\frac{b_1}{\beta}\right], \tag{A11}$$

where we have set

$$h\equiv\sum_k w(k)\sum_q w(k-q/2)^2\sum_{j,j'=1}^{2}(-1)^{j+j'+1}$$
$$\times\mathbf{M}_0^{-1}(k)_{2j}\mathbf{M}_0^{-1}(q-k)_{j'j}\langle\underset{\sim}{\mathbf{b}}(q)\underset{\sim}{\mathbf{b}}^\dagger(q)\rangle_{S_{\mathrm{eff}}^{(2)}}^{j'j}\mathbf{M}_0^{-1}(k)_{j'1} \tag{A12}$$

and made use of Eqs. (2.37) and (2.38). Upon manipulating the derivatives in Eq. (A4), it can be finally shown that the expression within brackets on the right-hand side of Eq. (A11) vanishes identically. Equation (A11) thus reduces to $\Delta-\Delta_0\equiv\Delta_1=b_1/\beta$, as expected.

We remark finally that, when $\Delta_0=0$, Eq. (A11) reduces to $A(q=0)b_1=0$, since in this case $h=0$ identically. This implies that $\Delta_1=0$, too.

APPENDIX B: MOMENTUM DEPENDENCE OF THE INTERACTION POTENTIAL FOR COMPOSITE BOSONS

In Sec. III we have mapped the original fermionic system interacting via an attractive-potential onto an effective system of composite bosons, in the limit of strong fermionic attraction. We have also determined the ''strength'' of the effective *residual* interactions among the composite bosons, which led us to conclude that retaining only the quartic interaction is sufficient to describe the bosonization limit. In this Appendix we study the *momentum* dependence of the quartic interaction, from which we will conclude that ξ_{pair} identifies the characteristic length scale of the boson-boson interaction.

To this end, it is convenient to simplify the expression (3.15) by setting $w(\mathbf{k}=1)$, that corresponds to taking $k_0=\infty$ from the outset in Eq. (2.12). It is then clear from dimensional analysis of Eq. (3.15), together with Eqs. (3.8)–(3.11) in the limits $k_0=\infty$ and $\beta\mu\to-\infty$, that $|\mu|$ and $(2m|\mu|)^{-1/2}$ constitute the only energy and length scales in the problem, respectively. For scattering processes among the composite bosons which involve (Matsubara) frequencies *small* compared to $|\mu|$, one can thus set all (external) bosonic frequencies equal to zero in Eq. (3.15) because to this limit there corresponds a well-defined *finite* value of the interaction potential, as we have verified in Sec. III. We shall consistently not be particularly interested in the frequency dependence of the effective boson-boson potential. Regarding instead its momentum dependence, we would expect a *truly* bosonic potential to be cast in the (symmetrized) form:

$$v_2(q_1\cdots q_4)=\delta_{q_1+q_2,q_3+q_4}[u(\mathbf{q}_1-\mathbf{q}_3)+u(\mathbf{q}_1-\mathbf{q}_4)], \tag{B1}$$

$u(\mathbf{q})$ being the Fourier transform of the two-body interaction potential. In fact, we shall verify below that Eq. (B1) holds *approximately* only for $|\mathbf{q}_i|(2m|\mu|)^{-1/2}\ll1(i=1,\ldots,4)$ with $u(\mathbf{q})$=const. In other words, the residual boson-boson potential can itself be approximated by a ''contact'' potential *provided* only small-momentum scattering processes are considered.

To verify to what extent Eq. (B1) is valid, we consider explicitly two degenerate cases with (i) $q_1=q_2=q_3=q_4=q$ and (ii) $q_1=-q_2=q$ and $q_3=q_4=0$, for which Eq. (B1) would give $v_2(q,q,q,q)=2u(\mathbf{q}=0)$ and $v_2(q,-q,0,0)=2u(\mathbf{q})$, respectively. In the first case, we obtain for the (four) momentum sum in Eq. (3.15) (in three dimensions):

$$\sum_k\frac{1}{\epsilon(k)^2\epsilon(q-k)^2}=\frac{1}{16\pi}\,\beta\Omega\,\frac{(2m|\mu|)^{3/2}}{(2|\mu|)^3}$$
$$\times\left[1+\left(\frac{\mathbf{q}^2}{4m}-i\omega_\nu\right)\frac{1}{2|\mu|}\right]^{-3/2}. \tag{B2}$$

In the second case we obtain instead

$$\sum_k \frac{1}{\epsilon(k)\epsilon(-k)\epsilon(k-q)\epsilon(q-k)} = \frac{\beta}{2}\sum_{\mathbf{k}} \frac{1}{\xi_{\mathbf{k}}[\xi^2_{\mathbf{k}-\mathbf{q}}-(i\omega_\nu-\xi_{\mathbf{k}})^2]}+\text{c.c.}, \tag{B3}$$

which for $\omega_\nu=0$ and in three dimensions reduces to

$$\frac{1}{\pi}\beta\Omega\frac{(2m|\mu|)^{3/2}}{(2|\mu|)^3}\frac{1}{\tilde{q}^2(4+\tilde{q}^2)} \times\left[(4+\tilde{q}^2)^{1/2}-1-\frac{2}{\pi}\arctan(\tilde{q}/2)-\frac{2}{\pi}\arctan(2/\tilde{q})\right] \tag{B4}$$

with $\tilde{q}=|\mathbf{q}|(2m|\mu|)^{-1/2}$. The desired values $v_2(q,q,q,q)$ and $v_2(q,-q,0,0)$ are obtained eventually upon dividing the results (B2) and (B3), respectively, by $|A'(q)|^2$ and $|A'(q)|A'(0)$.

It is clear from the definition (3.11) of $A'(q)$ [together with (3.8) of $A(q)$] that its computation requires a suitable (ultraviolet) regularization of the momentum integral when $w(\mathbf{k})=1$. We follow here a standard procedure in the literature and introduce the scattering amplitude a_s defined via the equation[7,15]

$$\frac{m}{4\pi a_s}=\frac{1}{\Omega V}+\frac{1}{\Omega}\sum_{\mathbf{k}}\frac{m}{\mathbf{k}^2} \tag{B5}$$

in the center-of-mass reference frame of the two fermions. The (ultraviolet) divergent sum on the right-hand side of Eq. (B5) results in a finite value of a_s by letting $V\to 0$ in a suitable way. $A(\mathbf{q},i\omega_\nu)$ given by Eq. (3.8) becomes accordingly

$$\frac{A(\mathbf{q},i\omega_\nu)}{\beta\Omega}=\frac{1}{\Omega}\sum_{\mathbf{k}}\left[\frac{m}{\mathbf{k}^2}-\left(\frac{\mathbf{k}^2}{m}+\frac{\mathbf{q}^2}{4m}-2\mu-i\omega_\nu\right)^{-1}\right] -\frac{m}{4\pi a_s}. \tag{B6}$$

Solution of Eq. (3.9), in turn, yields

$$\frac{1}{\Omega}\sum_{\mathbf{k}}\left[\frac{m}{\mathbf{k}^2}-\left(\frac{\mathbf{k}^2}{m}+\epsilon_0\right)^{-1}\right]-\frac{m}{4\pi a_s}=0 \tag{B7}$$

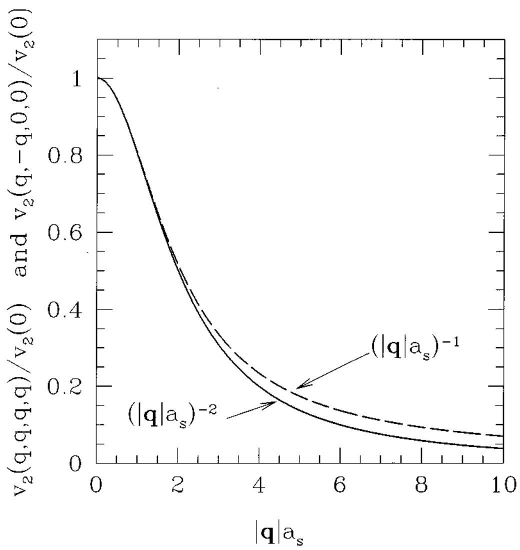

FIG. 10. Graphical representation of Eqs. (B9) (dashed line) and (B10) (full line) vs $\tilde{q}=|\mathbf{q}|a_s$. The characteristic decay of the two functions for large $\tilde{q}$ is indicated.

with ϵ_0 defined via Eq. (3.10). In this way we obtain from Eq. (3.11)

$$\frac{A'(\mathbf{q},i\omega_\nu)}{\beta\Omega}=\frac{1}{\Omega}\sum_{\mathbf{k}}\left(\frac{\mathbf{k}^2}{m}+\epsilon_0\right)^{-1}\left(\frac{\mathbf{k}^2}{m}+\frac{\mathbf{q}^2}{4m}-2\mu-i\omega_\nu\right)^{-1} =\frac{1}{4\pi}\frac{m^{3/2}}{\epsilon_0^{1/2}}\frac{[1+(\mathbf{q}^2/4m-i\omega_\nu)1/2|\mu|]^{1/2}-1}{(\mathbf{q}^2/2m-i\omega_\nu)1/2|\mu|}, \tag{B8}$$

where the last equality holds in three dimensions and $2\mu=-\epsilon_0$ within our approximations.

According to Eqs. (3.15) and (3.16), we obtain eventually for $\omega_\nu=0$ (in three dimensions)

$$\frac{v_2(\mathbf{q},\mathbf{q},\mathbf{q},\mathbf{q})}{v_2(0)}=\frac{1}{2}\frac{\tilde{q}^4}{(4+\tilde{q}^2)^{3/2}(\sqrt{4+\tilde{q}^2}-2)^2} \tag{B9}$$

and

$$\frac{v_2(\mathbf{q},-\mathbf{q},0,0)}{v_2(0)}=\frac{4[(4+\tilde{q}^2)^{1/2}-1-(2/\pi)\arctan(\tilde{q}/2)-(2/\pi)\arctan(2/\tilde{q})]}{(4+\tilde{q}^2)(\sqrt{4+\tilde{q}^2}-2)} \tag{B10}$$

with $\tilde{q}$ defined after Eq. (B4). The behavior of the expressions (B9) and (B10) versus $\tilde{q}$ is depicted in Fig. 10. Since $\tilde{q}$ can be also written as the product $|\mathbf{q}|a_s$, from Eqs. (B9) and (B10) we conclude that (i) $v_2(\mathbf{q},\mathbf{q},\mathbf{q},\mathbf{q})=2u(\mathbf{q}=0)=$constant can be approximately true only for $|\mathbf{q}|\ll a_s^{-1}$, while $v_2(\mathbf{q},\mathbf{q},\mathbf{q},\mathbf{q})$ decays as $(|\mathbf{q}|a_s)^{-1}$ for $|\mathbf{q}|\gg a_s^{-1}$; (ii) the dependence of $v_2(\mathbf{q},-\mathbf{q},0,0)$ on $|\mathbf{q}|$ is (approximately) given by $v_2(0)4a_s^{-2}/(\mathbf{q}^2+4a_s^{-2})$. These results imply that the composite nature of the bosons prevents Eq. (B1) from holding strictly for ''large'' momenta (and energies). Nevertheless, our finding that $v_2(\mathbf{q},-\mathbf{q},0,0)$ decays more rapidly than $v_2(\mathbf{q},\mathbf{q},\mathbf{q},\mathbf{q})$ for $|\mathbf{q}|\gg a_s^{-1}$ makes assumption (B1) valid in a ''weak'' sense.

A final comment is in order. Although we have introduced the scattering amplitude a_s via Eq. (B5) to comply with a standard procedure in the literature, it is clear that there exists a *single* characteristic length in the bosonic limit, which (in three dimensions) can be identified alternatively with a_s, $\xi_{\rm pair}$, or r_0 (r_0 being the bound-state radius for the associated two-fermion problem), the three lengths differing at most by numerical constants of order unity. With the ''contact'' potential $V\Omega\,\delta(\mathbf{r})$ adopted in this Appendix, in fact, we readily find for the solution of the two-fermion Schrödinger equation in momentum space:

$$\phi(\mathbf{k}) = -\frac{V\sqrt{\Omega}\,\phi(0)}{\mathbf{k}^2/m+\epsilon_0}, \tag{B11}$$

where

$$\phi(0) \equiv \frac{1}{\sqrt{\Omega}}\sum_{\mathbf{k}}\ \phi(\mathbf{k}) = -V\phi(0)\sum_{\mathbf{k}}\ \frac{1}{\mathbf{k}^2/m+\epsilon_0} \tag{B12}$$

plays the role of the bound-state equation. Eliminating V in favor of a_s via Eq. (B5), we find eventually (in three dimensions):

$$\epsilon_0 = \frac{1}{m a_s^2}. \tag{B13}$$

By the same token, the bound-state radius r_0 is given by

$$r_0^2 = \frac{\Sigma_{\mathbf{k}}|\nabla_{\mathbf{k}}\phi(\mathbf{k})|^2}{\Sigma_{\mathbf{k}}|\phi(\mathbf{k})|^2} = \frac{a_s^2}{2}, \tag{B14}$$

where the last equality holds in three dimensions. Since we also know that r_0 coincides with $\xi_{\rm pair}$ in the BE limit,[24] the results of this Appendix could be expressed in terms of $\xi_{\rm pair}$ instead of a_s. This procedure will consistently be adopted in Appendix C also in the BCS regime. Our preference for $\xi_{\rm pair}$ over a_s stands from the fact that $\xi_{\rm pair}$ is (at least, in principle) experimentally accessible, since it pertains to the physical problem of interest (while a_s is a fictitious parameter of the theory). Besides, we shall find in Appendix C that expressing the relevant physical quantities in terms of $\xi_{\rm pair}$ from the outset requires no explicit regularization of divergent expressions.

APPENDIX C: $\xi_{\rm phase}$ VS $\xi_{\rm pair}$ FOR A CONTACT POTENTIAL

In the text we have adopted a fermionic interaction potential of the (separable) form (2.11) with $w(\mathbf{k})$ given by Eq. (2.12). By doing so, we have introduced an intrinsic length scale (k_0^{-1}) for the potential, which we have exploited to simplify the regularization procedure and to explore the density dependence of the results. This additional flexibility has enabled us to verify the independence from k_0 of the relevant results in the BCS limit, although physical restrictions limit k_0 to ''large'' values (cf. Refs. 30 and 43). For this reason we have sometimes considered in the text the limit $k_0\to\infty$ for the final expressions, where they get considerably simplified (see also Ref. 44).

The purpose of this Appendix is to study *directly* the case $k_0=\infty$, for which $w(\mathbf{k})=1$ and the interaction reduces to a ''contact'' potential in real space. This potential has already been considered in Appendix B to simplify the calculations; in that context, we have adopted a standard regularization procedure in terms of the scattering amplitude a_s. Here we shall avoid introducing a_s and use $\xi_{\rm pair}$ instead. Setting $k_0=\infty$ from the outset will also make some approximations used in the text for analytic calculations more transparent.

We begin by evaluating $\xi_{\rm pair}$ according to Eqs. (2.8) and (2.9) in d dimensions:

$$\xi_{\rm pair}^2 = \frac{\Sigma_{\mathbf{k}}|\nabla_{\mathbf{k}}\phi(\mathbf{k})|^2}{\Sigma_{\mathbf{k}}|\phi(\mathbf{k})|^2} = \frac{1}{m^2}\frac{\int_0^\infty dk(k^{d+1}\xi_k^2/E_k^6)}{\int_0^\infty dk(k^{d-1}/E_k^2)}, \tag{C1}$$

where now $\phi(\mathbf{k}) = 1/E_{\mathbf{k}} = (\xi_{\mathbf{k}}^2+\Delta_0^2)^{-1/2}$. Here $\xi_{\rm pair}$ is a function of μ and Δ_0 only. Δ_0, in turn, can be related to μ via the (zero-temperature mean-field expression of the) number density (or, alternatively, via k_F):

$$n \equiv \frac{2}{d}\,K_d k_F^d = K_d\int_0^\infty k^{d-1}\left(1-\frac{\xi_k}{E_k}\right) \tag{C2}$$

$(2\pi)^d K_d$ being the area of the unit sphere in d dimensions.

The expression of $\xi_{\rm phase}$ is still given by Eq. (2.62), where now

$$a = \frac{\Delta_0^2 K_d}{2}\int_0^\infty dk\,\frac{k^{d-1}}{E_k^3} \tag{C3}$$

and

$$b = \frac{K_d}{16m}\int_0^\infty dk\,\frac{k^{d-1}\xi_k^2}{E_k^5}\left[\frac{(\xi_k^2-2\Delta_0^2)}{\xi_k}+\frac{5\Delta_0^2}{dm}\,\frac{k^2}{E_k^2}\right]. \tag{C4}$$

Note that the interaction strength V does not appear explicitly in Eqs. (C1)–(C4) (this remains true even for $w(\mathbf{k})\neq$const).

Inversion of Eqs. (C1) and (C2) yields μ and Δ_0 as functions of k_F and $\xi_{\rm pair}$, without invoking the gap equation (2.33). The lack of an intrinsic length (such as k_0^{-1}) in the potential enables us to write further:

$$\mu = \mu(k_F,\xi_{\rm pair}) = \frac{k_F^2}{2m}\,h^{(\mu)}(k_F\xi_{\rm pair}) \tag{C5}$$

and

$$\Delta_0 = \Delta_0(k_F,\xi_{\rm pair}) = \frac{k_F^2}{2m}\,h^{(\Delta_0)}(k_F\xi_{\rm pair}), \tag{C6}$$

where $h^{(\mu)}$ and $h^{(\Delta_0)}$ are functions of the dimensionless variable $k_F\xi_{\rm pair}$ only (this is not true, however, when $w(\mathbf{k})\neq$const). As a consequence, we write from Eqs. (C3) and (C4):

$$k_F\xi_{\rm phase} = h^{(\xi)}(k_F\xi_{\rm pair}) \tag{C7}$$

where $h^{(\xi)}$ is an additional function of $k_F\xi_{\rm pair}$ only.

Quite generally, Eqs. (C5)–(C7) can be solved numerically (in spatial dimensions $d<4$) for any desired value of $k_F\xi_{\rm pair}$. This procedure has been used in Sec. IV to determine the limiting curves for $k_0=\infty$ as well as the dependence of $\xi_{\rm phase}$ on dimensionality. In the rest of this Appendix we discuss the analytic BCS and BE limits.

We consider first the BCS limit and note that the integrals in Eqs. (C1)–(C4) can be cast in the form

$$I_n^m(\mu,\Delta_0)=\int_{-\mu}^{+\infty} d\xi\, H(\xi)\, \frac{\xi^m}{E(\xi)^n}, \tag{C8}$$

where (m,n) are nonnegative integers and $H(\xi)$ is a smooth function of ξ which, by assumption, does not spoil the ultraviolet convergence of the integral and remains finite for $\xi\to 0$. In the BCS limit, $\Delta_0\to 0$ and the integral (C8) develops an infrared singularity when $n-m\geqslant 1$. Equation (C8) is then manipulated as follows:

$$I_n^m(\mu,\Delta_0)=H(0)\int_{-\mu}^{+\infty} d\xi\, \frac{\xi^m}{E(\xi)^n} + \int_{-\mu}^{+\infty} d\xi \left(\frac{H(\xi)-H(0)}{\xi}\right) \frac{\xi^{m+1}}{E(\xi)^n}, \tag{C9}$$

where now the first integral on the right-hand side can be evaluated analytically, yielding

$$H(0)\Delta_0^{m-n+1}\int_{-\mu/\Delta_0}^{+\infty} dy\, \frac{y^m}{(y^2+1)^{n/2}} \cong H(0)\Delta_0^{m-n+1} J_n^m \tag{C10}$$

with

$$J_n^m \equiv \int_{-\infty}^{+\infty} dy\, \frac{y^m}{(y^2+1)^{n/2}} \tag{C11}$$

since $\mu/\Delta_0\to\infty$ in the BCS limit. Concerning the second integral on the right-hand side of Eq. (C9), it may or may not converge in the infrared when $\Delta_0\to 0$. If it does converge, this term can be safely neglected in comparison to (C10) for $n-m>1$; otherwise, the procedure followed in Eq. (C9) can be iterated for the function $H'(\xi)=[H(\xi)-H(0)]/\xi$ in the place of $H(\xi)$, until the resulting integral converges. In any event, the terms generated in this way are subleading with respect to (C10) as $\Delta_0\to 0$, and can be neglected in the limit. The same procedure has been used in the text to obtain the results (2.66) and (2.67).

With the above approximations, we obtain from the leading terms of Eqs. (C1), (C3), and (C4) in the BCS limit:

$$(\xi_{\text{pair}}^{\text{BCS}})^2=\frac{2\mu}{m\Delta_0^2}\frac{J_6^2}{J_2^0}=\frac{\mu}{4m\Delta_0^2}, \tag{C12}$$

$$(\xi_{\text{phase}}^{\text{BCS}})^2=\frac{5\mu}{4dm\Delta_0^2}\frac{J_7^2}{J_3^0}=\frac{\mu}{6dm\Delta_0^2}, \tag{C13}$$

which recover Eqs. (2.69) and (2.68), respectively, in the limit $k_0\to\infty$ and $d=3$. We can thus write in the BCS limit

$$k_F\xi_{\text{phase}}^{\text{BCS}}=\sqrt{\frac{2}{3d}}\, k_F\xi_{\text{pair}}^{\text{BCS}}. \tag{C14}$$

In the BE limit, on the other hand, the approximation $\mu/\Delta_0\to -\infty$ applies and the integrals in (C1)–(C4) are conveniently evaluated by expanding their integrands in power of $\Delta_0/|\mu|$. One obtains to leading order

$$(\xi_{\text{pair}}^{\text{BE}})^2=\frac{2}{m|\mu|}\frac{I_1(d)}{I_2(d)}, \tag{C15}$$

$$n=\frac{K_d}{2}(2m)^{d/2}\Delta_0^2|\mu|^{d/2-2}I_2(d), \tag{C16}$$

$$(\xi_{\text{phase}}^{\text{BE}})^2=\frac{|\mu|}{8m\Delta_0^2}\frac{I_2(d)}{I_3(d)}, \tag{C17}$$

with $(d<4)$

$$I_1(d)\equiv\int_0^\infty dy\, \frac{y^{d+1}}{(y^2+1)^4}=\frac{\Gamma[(d+2)/2]\Gamma[(6-d)/2]}{12}, \tag{C18a}$$

$$I_2(d)\equiv\int_0^\infty dy\, \frac{y^{d-1}}{(y^2+1)^2}=\frac{\Gamma(d/2)\Gamma[(4-d)/2]}{2}, \tag{C18b}$$

$$I_3(d)\equiv\int_0^\infty dy\, \frac{y^{d-1}}{(y^2+1)^3}=\frac{\Gamma(d/2)\Gamma[(6-d)/2]}{4}, \tag{C18c}$$

Γ being the Euler's gamma function. Expressing $|\mu|$ in terms of $\xi_{\text{pair}}^{\text{BE}}$ from (C15) and Δ_0 in terms of n (and thus of k_F) from (C16), and entering the results into Eq. (C17), we obtain in the BE limit

$$(k_F\xi_{\text{phase}}^{\text{BE}})^2=\frac{d}{16}\frac{I_2(d)^2}{I_3(d)}\left(\frac{4I_1(d)}{I_2(d)}\right)^{d/2-1}(k_F\xi_{\text{pair}}^{\text{BE}})^{2-d}. \tag{C19}$$

Note that for $d=3$ this expression coincides with the result obtained previously in the limit $k_0\to\infty$ (cf. Ref. 44). Note also that, contrary to the BCS result (C14) which depends weakly on d, the BE expression (C19) depends markedly on d and shows a peculiar behavior for $d\to 2$.

APPENDIX D: BEHAVIOR OF THE CHEMICAL POTENTIAL VS $k_F\xi_{\text{pair}}$

In Ref. 24 it was found that the crossover between the BCS and BE regimes occurs in a rather *narrow* range of the parameter $k_F\xi_{\text{pair}}$, by examining the behavior of the chemical potential vs $k_F\xi_{\text{pair}}$ at the mean-field level. This finding has been confirmed in the present paper by looking at the behavior of $k_F\xi_{\text{phase}}$ vs $k_F\xi_{\text{pair}}$ with the inclusion of fluctuations. The purpose of this Appendix is to investigate to what extent the behavior of the chemical potential vs $k_F\xi_{\text{pair}}$ is ''universal,'' in the sense that it is sufficiently independent from the specific model Hamiltonian and from the dimensionality (at least at the mean-field level).

To this end, we shall examine (i) the continuum model Hamiltonian (2.10)–(2.12) in the limit $k_0=\infty$ (''contact'' potential), for which simplifications occur, at intermediate values of the dimensionality $(2\leqslant d\leqslant 3)$; (ii) the negative-$U$ Hubbard model on a cubic $(d=3)$ lattice.

The equations determining the chemical potential μ and the gap parameter Δ_0 vs $k_F\xi_{\text{pair}}$ for $k_0=\infty$ and intermediate dimensionality are reported in Appendix C [cf., in particular, Eqs. (C1) and (C2)]. Their numerical solution yields the be-

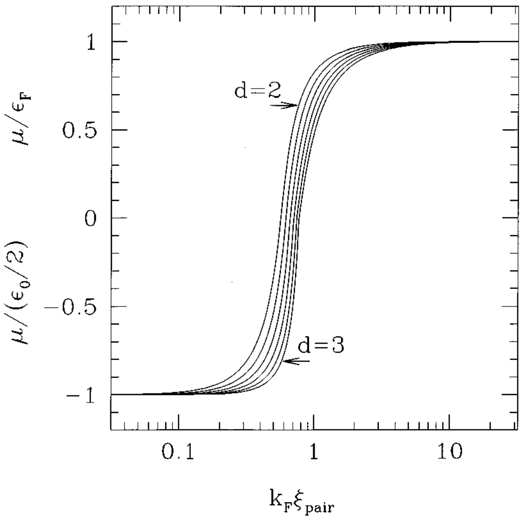

FIG. 11. Chemical potential μ vs $k_F\xi_{pair}$ (at zero temperature) for $k_0=\infty$ and $2\leqslant d\leqslant 3$. Different curves are labeled by the values of d (in steps of 0.2). Positive values of μ are normalized by the Fermi energy $\epsilon_F=k_F^2/2m$, while negative values of μ by half the magnitude ϵ_0 of the eigenvalue of the two-body problem in d dimensions.

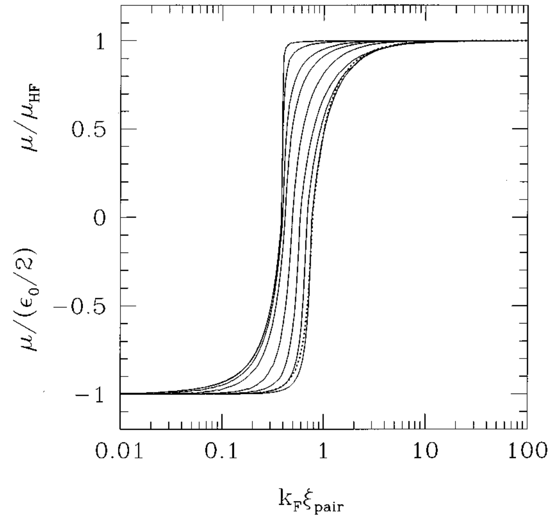

FIG. 12. Chemical potential vs $k_F\xi_{pair}$ for the $d=3$ negative-U Hubbard model at the mean-field level. Values of band filling label different curves ($f=0.01$, 0.1, 0.2, 0.3, 0.4, 0.45, and 0.49 from right to left). Energies are measured from the bottom of the single-particle band. The curve with $d=3$ from Fig. 11 is shown for comparison (dotted line). Normalization of μ is as in Fig. 11, except for the replacement of the Fermi energy by the Hartree-Fock chemical potential μ_{HF}.

havior of μ vs $k_F\xi_{pair}$ shown in Fig. 11 for $2\leqslant d\leqslant 3$. The curve for $d=3$ coincides with the curve reported in Fig. 1 of Ref. 24 in the limit of low reduced density, and the curve for $d=2$ coincides with the two-dimensional analytic results given in Ref. 2 (once expressed in terms of $k_F\xi_{pair}$). Note also from Fig. 11 that, at the mean-field level, there is no significant difference between the results for $d=2$ and $d=3$.

The crossover from BCS to BE for the $d=3$ negative-U Hubbard model was originally discussed in Ref. 11 in terms of the interaction strength U. Here we repeat this mean-field calculation, by taking $k_F\xi_{pair}$ (in the place of U) as the variable driving the crossover. The calculation proceeds similarly to that for the continuum model, *but* for the additional inclusion of (normal-state) Hartree-Fock terms in the mean-field decoupling.[38] These terms are now relevant since they provide a sizable shift of the chemical potential near half filling of the electronic band, where they signal the occurrence of a liquid-gas phase separation through a nonmonotonic behavior of the chemical potential vs band filling. As the inclusion of pairing restores the correct increase of the chemical potential with filling, a Maxwell construction is required to determine the normal-state value of the chemical potential only.

The chemical potential vs $k_F\xi_{pair}$ for the $d=3$ negative-U Hubbard model is shown in Fig. 12 for several band fillings. Also shown for comparison is the curve for $d=3$ reproduced from Fig. 11 (dotted line). The comparison evidences the peculiar behavior of the Hubbard model near half filling ($f=1/2$), while the continuum-model results are recovered in the low-density limit ($f\ll 1$). Note that the qualitative behavior for the Hubbard model looks similar to that for the continuum model even at intermediate fillings.

Notwithstanding these similarities, a warning on the nature of the bosonic limit for the negative-U Hubbard model is in order. Contrary to what happens for the continuum model (or else, in the low-density limit $f\ll 1$), the gap equation does not reduce to the bound-state equation for the two-fermion problem when $k_F\xi_{pair}\ll 1$, since near half filling one finds $\Delta=|\mu|$ and the condition $\Delta\ll|\mu|$ cannot be satisfied. As a consequence, the broken-symmetry state is not a BE condensate in the conventional sense, as discussed in Ref. 11.

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[10] The crossover of ξ_{phase} from BCS to BE has been briefly discussed in Ref. 7, expanding the action of the functional integral near the critical temperature where the gap parameter is small. We have considered instead the broken-symmetry state at zero temperature directly. For this reason, our analysis should be more reliable and exhaustive.

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[18] As a consequence, satisfying the conservation laws for the *effective* bosonic system resulting from a superconducting fermionic system in the extreme strong-coupling limit poses no problem (cf. Ref. 15).

[19] An extensive literature exists on the *dilute* Bose gas. For recent reviews, see: P. Nozières and D. Pines, *The Theory of Quantum Liquids* (Addison-Wesley, Redwood City, CA, 1990), Vol. II; A. Griffin, *Excitations in a Bose-Condensed Liquid* (Cambridge University Press, Cambridge, 1993).

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[21] Although a functional-integral formulation has been already used in Ref. 7 to recover the NSR crossover from BCS to Bose superconductivity, apparently the need of going beyond Gaussian fluctuations to be consistent with the loop expansion has not been pointed out in that paper.

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[23] A detailed proof of this statement will be given elsewhere.

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[25] The condition $k_F\xi_{\mathrm{pair}}\simeq 10$ has been identified as the beginning of the crossover from BCS to BE also by an approach based on Eliashberg theory for superconductivity [G. Varelogiannis and L. Pietronero (unpublished)].

[26] In a recent paper by M. Casas *et al.* [Phys. Rev. B **50**, 15 945 (1994)] the coherence length for two-electron correlation (namely, ξ_{pair} of the present paper) was calculated at the mean-field level for a momentum-independent gap energy and compared with available experimental data on cuprate superconductors. Since the data refer to ξ_{phase} instead, a comparison of this kind is meaningful as far as $\xi_{\mathrm{phase}}\simeq\xi_{\mathrm{pair}}$, which we show in the present paper to hold provided $k_F\xi_{\mathrm{pair}}\gtrsim 10$.

[27] See, e.g., D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Benjamin, MA, 1975), Chaps. 10 and 12; see also P. B. Weichman, Phys. Rev. B **38**, 8739 (1988).

[28] We set $\hbar=1$ throughout.

[29] A Hamiltonian similar to (2.10) results from the so-called negative-U Hubbard model in a lattice, whereby $V(\mathbf{k},\mathbf{k}')$ is replaced by the constant U, $\xi_{\mathbf{k}}$ by the band dispersion relation, and the wave vectors are restricted to the Brillouin zone. In Appendix D we consider this model Hamiltonian at the mean-field level.

[30] The Hamiltonian (2.10) is, in general, invariant under a global but not a local gauge transformation. Local gauge invariance is suitably recovered whenever the interaction potential $V(\mathbf{k},\mathbf{k}')$ depends weakly on $\mathbf{k}$ and $\mathbf{k}'$, i.e., in the limit $k_0\to\infty$ for the choices (2.11) and (2.12).

[31] For a recent pedagogical review on how path integrals can be written for fermions, see R. Shankar, Rev. Mod. Phys. **66**, 129 (1994), Sec. III, and references quoted therein.

[32] The separable potential (2.11) has led us to represent the interaction part of the Hamiltonian as a bilinear form in the operators (2.17). When $w(\mathbf{k})$ depends weakly on $\mathbf{k}$, an alternative decoupling in terms of the density-fluctuation operators $\rho_\sigma(\mathbf{q},\tau)=\Sigma_{\mathbf{k}}\overline{c_\sigma}(\mathbf{k}+\mathbf{q},\tau)c_\sigma(\mathbf{k},\tau)$ is also possible. As discussed in Ref. 5, linear responses for the two operators $\rho_\sigma(\mathbf{q})$ and $\mathscr{B}(\mathbf{q})$ are, quite generally, coupled. In the following, we shall not consider this coupling at the Gaussian level since it is not essential for our purposes.

[33] To be more precise, the time label that specifies $\overline{\mathscr{B}}(\mathbf{q})$ should be augmented by a positive infinitesimal ($\tau\to\tau+\delta$ with $\delta=0^+$) on both sides of Eq. (2.18). This implies that the Fourier-transformed variables have to be interpreted (in the mixed fermionic-bosonic terms only) as $b(\mathbf{q},\omega_\nu)\to b(\mathbf{q},\omega_\nu)e^{i\omega_\nu\delta}$ and $b^*(\mathbf{q},\omega_\nu)\to b^*(\mathbf{q},\omega_\nu)$.

[34] P. B. Weichman, Phys. Rev. B **38**, 8739 (1988).

[35] In other words, the identity $f(\lambda)=0$ entails the vanishing of every coefficient f_n of the expansion $f(\lambda)=\Sigma_{n=0}^{\infty}f_n\lambda^n$.

[36] The standard BCS expression for the grand-canonical free energy at temperature β^{-1}, namely, $F_0^{(\mathrm{BCS})}=-|\Delta_0|^2/V-(2/\beta)\Sigma_{\mathbf{k}}\ln(1+e^{-\beta E_{\mathbf{k}}})+\Sigma_{\mathbf{k}}(\xi_{\mathbf{k}}-E_{\mathbf{k}})$, results from Eq. (2.35) *provided* the sum over the fermionic frequencies ω_s (which is implicit in the trace therein) is evaluated carefully by the correct time discretization procedure of the fermionic functional integral.

[37] In Eqs. (2.37) and (2.38) the symmetry properties of $A(q)$ and $B(q)$ when $q\to-q$ hold provided $w(-\mathbf{k})=w(\mathbf{k})$.

[38] Hartree-Fock terms do not appear in the expression of F_0 owing to our mean-field decoupling; they appear instead in the expression of F_1, as one can readily verify for the ''normal'' ($\Delta_0=0$) phase by expanding F_1 in powers of V.

[39] E. Arrigoni, C. Castellani, M. Grilli, R. Raimondi, and G. C. Strinati, Phys. Rep. **241**, 291 (1994).

[40] Besides the diagrams depicted in Fig. 3, the ''exchange'' contribution (2.48) contains also terms with the structure of Eq. (2.53), where the fermionic bubbles (A and B) are calculated with the ''dressed'' version of the single-particle fermionic Green's functions that include corrections of order λ [cf. the first term on the right-hand side of Eq. (2.50)]. Despite this replacement, this contribution is expected to maintain the ''short-range'' character of the mean-field expression (2.53) and is accordingly neglected in the following.

[41] More precisely, the characteristic spatial ''range'' associated with a (non-negative) function $F(\mathbf{r})$, whose Fourier transform $F(\mathbf{q})$ is well defined, can be determined by

$$\langle r^2\rangle\equiv\frac{\int d\mathbf{r}\,F(\mathbf{r})\mathbf{r}^2}{\int d\mathbf{r}\,F(\mathbf{r})}=\frac{-\nabla_{\mathbf{q}}^2F(\mathbf{q})|_{\mathbf{q}=0}}{F(\mathbf{q}=0)}.$$

This justifies retaining the quadratic expansion (2.61) only for the calculation of ξ_{phase}.

[42] The BCS limit is identified by the condition $\Delta_{k_F}\ll\mu$ with $\mu>0$. In this limit one can assume, in addition, that $\Delta_{k_F}\ll k_0^2/2m$ for any given k_0 [k_0 being the characteristic wave vector of the interaction potential, cf. Eq. (2.12)], provided one takes the strength V sufficiently small [cf. Eq. (2.11)]. With these two conditions, one verifies that the nonvanishing contributions to expression (2.64) [other than the contribution (2.66)] are smaller than the dominant contribution (2.66) by factors $(\Delta_{k_F}/\mu)(\Delta_{k_F}/k_0^2/2m)$ or $(\Delta_{k_F}/k_0^2/2m)^2$. For the approximations

used to derive the right-hand side of Eqs. (2.66) and (2.67), see also Appendix C.

[43] The BE limit is achieved when the condition $\Delta_0 \ll |\mu|$ with $\mu<0$ is satisfied. In this limit, however, the characteristic wave vector k_0 of the interaction plays an important role (in contrast with the BCS limit discussed in Ref. 42, for which the value of k_0 is irrelevant). The point is that increasing the value $|V|$ of the interaction strength makes $|\mu|$ increase accordingly, and thus $|\mu|$ becomes eventually larger than $k_0^2/2m$ for any initial choice of k_0. Expression (2.64) for the coefficient b, in turn, depends crucially on the ratio $|\mu|/(k_0^2/2m)$ and different values are obtained depending on which of the two conditions ($|\mu| \ll k_0^2/2m$ or $|\mu| \gg k_0^2/2m$) is satisfied. We have thus to impose some *restrictions* on the bosonization procedure according to the following scheme. For given k_0, $|V|$ is increased to reach the bosonization condition $\Delta_0 \ll |\mu|$ with $\mu<0$, paying attention that $|\mu|$ remains much smaller than $k_0^2/2m$. If this is the case, there is no need to increase $|V|$ any further and the bosonization condition has *effectively* been achieved. Otherwise, if $|\mu|$ needs to become comparable to or even larger than $k_0^2/2m$ to reach the bosonization condition $\Delta_0 \ll |\mu|$ with $\mu<0$, the calculation might produce inconsistent results, such as negative values for the coefficient b. We shall thus impose the *further condition* $|\mu| \ll k_0^2/2m$ on the BE limit (which is equivalent to a low-density condition for the Bose gas). With this condition, one verifies that the terms of the expression (2.64) [other than the contribution (2.70)] are smaller than the dominant contribution (2.70) by factors $(\Delta_0/|\mu|)$ and/or $(\Delta_0/k_0^2/2m)$ and their powers.

[44] It is interesting to derive from Eqs. (2.78) and (2.79) the limiting value of $\xi_{\text{phase}}^{\text{BE}}$ for $k_0 \to \infty$, whereby the interaction potential (in real space) reduces to a ''contact'' potential. Implementing this limit requires one to keep $\xi_{\text{pair}}^{\text{BE}}$ constant, in such a way that $|\mu_0|$ remains also constant and $c \to 0$ in Eq. (2.79). One then obtains $(k_F \xi_{\text{phase}}^{\text{BE}})^2 = (3\pi/16)\sqrt{|\mu_0|/\epsilon_F}$ with $\epsilon_F = k_F^2/2m$. Alternatively, relating $|\mu_0|$ to $\xi_{\text{pair}}^{\text{BE}}$ (cf. Appendix C) one rewrites $(k_F \xi_{\text{phase}}^{\text{BE}})^2 = (3\pi/16\sqrt{2})(k_F \xi_{\text{pair}}^{\text{BE}})^{-1}$, which coincides (apart possibly for a numerical factor of order unity) with the result reported in Ref. 7 in terms of the scattering amplitude for the two-fermion problem.

[45] A related problem has been addressed in Ref. 15 using conventional diagrammatic techniques, where the two-fermion correlation function has been shown to reduce to the Bogolubov propagator in the strong-coupling limit. The functional-integral approach enables us to go further and study higher-order effects such as multiple-boson interactions.

[46] In the broken-symmetry state at zero temperature, the quadratic action (2.36) can be expanded in series of the small parameter Δ_0^2/μ^2 in the BE limit [cf. Eq. (3.24)]. In the absence of loops for the bosonic propagators, retaining the lowest significant order in Δ_0^2/μ^2 is equivalent to keeping $v_2(0)$ only out of the set $v_n(0)$ given by Eq. (3.29). In bosonic language, in fact, increasing n by one unit introduces in the self-energy two additional *condensate* lines proportional to $|\langle b'(q=0)\rangle|^2 \sim \beta\Omega|\mu|^{d/2}(\Delta_0/\mu)^2$ [cf. Eqs. (2.45), (3.13), and (3.18)].

[47] The bosonization criterion (3.22), which is exploited to obtain the effective bosonic action from the original fermionic action, ensures also that the resulting bosonic system is *dilute*. From the analysis of Appendix B one, in fact, obtains in three dimensions that the bound-state radius r_0 coincides with the scattering amplitude a_s for the two-fermion problem (apart from a numerical factor of order unity). Condition (3.22) is thus equivalent to the standard criterion $a_s^B n_B^{1/3} \ll 1$ for a dilute Bose gas, where $a_s^B = 2a_s$ and $n_B = n/2$.

[48] Cf., e.g., P. Nozières, in *Bose-Einstein Condensation*, edited by A. Griffin, D. W. Snoke, and S. Stringari (Cambridge University Press, Cambridge, 1995), p. 15.

[49] The coupling between longitudinal and transverse fluctuations has been formulated via the general principle of ''conservation of the modulus'' by A. Z. Patashinskii and V. L. Pokrovskii, Zh. Eksp. Teor. Fiz. **64**, 1445 (1973) [Sov. Phys. JETP **37**, 733 (1973)].

[50] Yu. A. Nepomnyashchii and A. A. Nepomnyashchii, Zh. Eksp. Teor. Fiz. **75**, 976 (1978) [Sov. Phys. JETP **48**, 493 (1978)].

[51] M. H. Anderson *et al.*, Science **269**, 198 (1995).