Kinetic equation for strongly disordered systems: Noninteracting electrons

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We consider the problem of casting the description of electronic transport, in the presence of strong disorder induced by impurities and in the absence of the electron-electron interaction, within the framework of the semiclassical Boltzmann transport equation which is familiar for weak disorder. A kinetic equation for noninteracting electrons acted upon by an external longitudinal field slowly varying in space and time is derived from the diagrammatic structure of the linear-response functions, in both cases of nonmagnetic and magnetic impurities. We show that this equation is of the Boltzmann type even in the presence of quantum-interference effects, with an effective scattering kernel which in both cases is nonlocal in time but for magnetic impurities is also nonlocal in space with a range of order of the mean free path. Implications for the physical mechanism leading to localization in the magnetic impurity case are discussed, and related to the lack of certain symmetry arguments when time-reversal invariance is absent. We also show how the anomalous scattering kernel leads to the more conventional description of localization phenomena in terms of a diffusion equation with a renormalized diffusion coefficient.

I. INTRODUCTION

Conventional transport theory of metals is based on the Boltzmann equation for conduction electrons, whereby the phase-space distribution function evolves along the flow lines between collisions with the impurities. The latter have the effect of introducing a microscopic time (the collision time $\tau$) and a microscopic length (the mean free path $l$) into the electronic propagation.$^1$ This approach is intrinsically a semiclassical one (the relevance of quantum mechanics being limited to the calculation of the scattering probability), and as such it can be recovered from a full quantum-mechanical approach by determining the equation of motion of the Wigner distribution function in the limit of dilute impurities.$^2$

In the theory of electronic localization one is interested, on the other hand, in the behavior of the electronic transport for times and distances that are large compared with their microscopic counterparts ($\tau$ and $l$). To this end, it is more natural to look directly at the diffusive aspect of the electronic transport which results into the hydrodynamic form of the correlation functions. In this approach calculations are usually implemented by the Kubo formalism within linear-response theory, and the localization transition is identified by the critical slowing down of the diffusion coefficient which results after the inclusion of quantum-interference effects over and above the Drude limit.$^3$

The question obviously arises whether localization effects can be reconciled with the Boltzmann transport theory. It was pointed out earlier that quantum-interference effects lead to violation of the standard Boltzmann transport description.$^4,5$ There has been, however, an increasing interest in the inclusion of localization effects into a kinetic description.$^2,6,7,8$ In particular, Hershfield and Ambegaokar have derived a Boltzmann-type equation in the presence of nonmagnetic impurities,$^6$ where the quantum-interference terms have been reabsorbed into an effective scattering kernel which is sharply enhanced in the backscattering direction. This is in agreement with the standard interpretation of localization in terms of quantum interference.$^9,4$ Their treatment relies on the Kadanoff and Baym approach to transport,$^{10}$ and thus is in principle not limited to small perturbations akin to the semiclassical Boltzmann equation.

In this paper we accomplish the derivation of a Boltzmann-type equation for noninteracting electrons in the presence of strongly disordered impurities with the use of the ordinary diagrammatic perturbation theory, which is usually adopted to treat localization problems$^3$ because of its direct relation with linear-response theory. In particular, we recover first the results by Hershfield and Ambegaokar$^6$ for the nonmagnetic impurity case and treat then the more involved magnetic impurity case for which the description of localization had so far no simple interpretation in terms of quantum interference. Specifically, we show that in the presence of magnetic impurities the effective scattering kernel of the Boltzmann equation becomes nonlocal in space within a distance of the order of the mean free path, yielding a reflection of the incident wave over a broad region of phase space. We find that the nonlocality of the scattering kernel is a distinctive feature of localization in the magnetic impurity case, being suppressed in the standard classical regime by the short range of the potential,$^{1,11}$ and in the nonmagnetic impurity case$^6$ by symmetry considerations that involve the interchange of the initial and final directions in the scattering processes owing to time reversal.
In interpreting any Boltzmann-type equation which includes quantum-interference effects one should be aware of the fact that the correspondence of its functional form with the semiclassical Boltzmann equation cannot be taken literally but should rather be interpreted in an effective way. That is to say, the inclusion of quantum interference requires one to lump at a given space point the effects of sequences of scattering events along paths much longer than \( l \). The important points are that an equation of the Boltzmann type can nonetheless be written and that this equation gives an intuitive account of the processes leading to localization. We limit ourselves in this paper to considering noninteracting electrons and defer the discussion of interaction effects to a following paper.\(^{13}\) A short summary of the work presented in this paper has been given in Ref. 13.

The plan of the paper is as follows. In Sec. II the choice of the distribution function is discussed and related to the calculation of a correlation function by linear-response techniques. The result is brought into the form of a Boltzmann equation for nonmagnetic and magnetic impurities in Secs. III and IV, respectively, where the inclusion of quantum-interference effects is shown to lead to quite different collision integrals in the two cases. The correspondence of the Boltzmann equation with the renormalized diffusion equation via a coarse-graining procedure is discussed in Sec. V. Section VI gives our conclusions.

II. CHOICE OF THE DISTRIBUTION FUNCTION FOR STRONGLY DISORDERED SYSTEMS AND ITS CALCULATION BY LINEAR-RESPONSE TECHNIQUES

The very definition of a phase-space distribution function \( n(\mathbf{k}; \mathbf{R}, T) \) requires that

\[
n(\mathbf{R}, T) = \int \frac{d\mathbf{k}}{(2\pi)^d} n(\mathbf{k}; \mathbf{R}, T)
\] (2.1)

gives the particle density at the spatial point \( \mathbf{R} \) and at time \( T \), and that

\[
j(\mathbf{R}, T) = \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{\mathbf{k}}{m} n(\mathbf{k}; \mathbf{R}, T)
\] (2.2)
gives the associated current density (\( m \) being the bare mass of a particle).\(^{14}\) In Eqs. (2.1) and (2.2) the integrals extend over all wave vectors \( \mathbf{k} \) in \( d \) dimensions. Although the distribution function \( n(\mathbf{k}; \mathbf{R}, T) \) can be strictly defined in the classical limit only, following Wigner\(^{15}\) it has been common practice to extend its definition to the quantum-mechanical case when the dependence on \( \mathbf{R} \) and \( T \) is slow compared to the characteristic microscopic quantities.

In the presence of disorder, the Wigner distribution function has to be averaged over the impurity configurations. This procedure introduces a broadening which affects the otherwise sharp drop of the unperturbed distribution at the Fermi surface. In the dilute impurities limit this broadening causes no harm, and the evolution of the distribution function is governed by the ordinary Boltzmann equation whereby each impurity scattering is treated as an independent event.\(^{2}\) Inclusion of quantum-interference effects (corresponding to weak localization) within a kinetic description is, however, prevented by the lack of momentum conservation associated with the broadening. One is thus led to consider alternative definitions for the phase-space distribution function that effectively overcome these difficulties.

To this end, we require the distribution function to satisfy the following conditions.

(i) It is real and fulfills Eqs. (2.1) and (2.2).

(ii) Besides the wave vector \( \mathbf{k} \), it introduces at an intermediate stage an additional dependence on a frequency \( \omega \) which is not affected by the presence of disorder.

(iii) It can be calculated by the ordinary diagrammatic perturbation theory which is usually adopted to treat localization problems,\(^{3}\) as well as to provide a microscopic basis for the Fermi liquid theory via the Landau-Boltzmann equation.\(^{16,17}\)

We thus introduce the following auxiliary function:

\[
G_{\omega}(\mathbf{k}; \mathbf{R}, T) = \int d\rho e^{-ik\cdot\rho} \int dt e^{i\omega t} \left[ \Theta(t) \left( \psi_{\sigma}^{\dagger} \begin{bmatrix} \mathbf{R} + \frac{\mathbf{P}}{2} + \frac{T}{2} \\ \frac{\mathbf{R} - \mathbf{P}}{2} + \frac{T}{2} \end{bmatrix} \psi_{\sigma}^{\dagger} \begin{bmatrix} \mathbf{R} - \frac{\mathbf{P}}{2} - \frac{T}{2} \\ \mathbf{R} + \frac{\mathbf{P}}{2} - \frac{T}{2} \end{bmatrix} \right) \\
- \Theta(-t) \left( \psi_{\sigma}^{\dagger} \begin{bmatrix} \mathbf{R} - \frac{\mathbf{P}}{2} - \frac{T}{2} \\ \mathbf{R} + \frac{\mathbf{P}}{2} + \frac{T}{2} \end{bmatrix} \psi_{\sigma}^{\dagger} \begin{bmatrix} \mathbf{R} - \frac{\mathbf{P}}{2} + \frac{T}{2} \\ \mathbf{R} + \frac{\mathbf{P}}{2} - \frac{T}{2} \end{bmatrix} \right) \right],
\] (2.3)

which coincides with the time-ordered single-particle Green's function in the "mixed representation" for the "slow" position \( \mathbf{R} \) and time \( T \) variables. In this expression \( \sigma \) is a spin label, \( \Theta(t) \) is the unit step function, and the time dependence of the field operators \( \Psi \) and \( \Psi^{\dagger} \) is understood within the Heisenberg picture that takes into account the time-dependent coupling of the system with an external agent.\(^{18}\) [In the zero-temperature limit we consider throughout, the quantum-mechanical average \( \langle \cdots \rangle \) on the right-hand side of Eq. (2.3) is taken over the unperturbed ground state.] In particular, the action of a longitudinal field can be represented by the interaction Hamiltonian

\[
\delta H(t) = \sum_{\sigma} \int d\mathbf{r} \psi_{\sigma}^{\dagger}(\mathbf{r}) \Phi_{\text{ext}}(\mathbf{r}, t) \psi_{\sigma}(\mathbf{r}).
\] (2.4)

To linear order in \( \Phi_{\text{ext}} \), the terms within parentheses on the right-hand side of Eq. (2.3) are obtained as follows:

\[
\delta G(1, 1') = i \int d2 F(1, 1'; 2) \Phi_{\text{ext}}(2),
\] (2.5)
where the labels \( 1, \ldots \) signify the set of space, time, and spin variables, with \( \sigma_1 = \sigma_2 = \sigma \) and
\[
t = t_1 - t_1, \quad \rho = r_1 - r_1, \quad T = \frac{1}{2}(t_1 + t_1), \quad R = \frac{1}{2}(r_1 + r_1).
\]
(2.6)
The kernel \( F \) of Eq. (2.5), namely,
\[
F(1, 1'; 2) = \Theta(t_1 - t_1') \Theta(t_1 - t_2) \langle [\Psi(1)\Psi(2), \Psi(1')] \rangle + \Theta(t_1 - t_1') \Theta(t_1' - t_2) \langle [\Psi(1)\Psi(2), \Psi(1')] \rangle
\]
\[
- \Theta(t_1' - t_1) \Theta(t_1' - t_2) \langle [\Psi(2)\Psi(1), \Psi(1')] \rangle - \Theta(t_1' - t_1) \Theta(t_1' - t_2) \langle [\Psi(1)\Psi(2), \Psi(1)] \rangle
\]
(2.7)
(where the time dependence of the field operators is now governed by the unperturbed Hamiltonian) can in turn be related to the two-particle correlation function
\[
L(1, 2; 1', 2') = \langle T[\Psi(1)\Psi(2)\Psi(1')\Psi(2')] \rangle - \langle T[\Psi(1)\Psi(1')] \rangle \langle T[\Psi(2)\Psi(2')] \rangle,
\]
(2.8)
where \( T \) is the time-ordering operator and \( 2' \) implies that the time variable \( t_2 \) is augmented by a positive infinitesimal. After Fourier transforming the "slow" variables \( R \) and \( T \) in Eq. (2.5) to the (small) external momentum \( Q \) and frequency \( \Omega \), we obtain
\[
\delta G_{10}(\mathbf{k}, \omega; Q, \Omega) = i\mathcal{F} \left[ \frac{k + Q}{2} + \frac{\omega + \Omega}{2} , -\frac{k + Q}{2} - \omega + \frac{\Omega}{2} \right] \Phi_{\text{ext}}(Q, \Omega)
\]
(2.9)
with the notation
\[
\mathcal{F}(q_1, v_1; q_1, v_1) = \int d(t_1 - t_1) d(t_1' - t_1') e^{-i(t_1 - t_1') q_1}
\]
\[
\times e^{-i(t_1' - t_1) q_1} \int d(t_1 - t_2) d(t_1' - t_2') e^{i(t_1 - t_2) v_1} e^{i(t_1' - t_2') v_1} \sum_{\sigma_2} F(1, 1'; 2).
\]
(2.10)
(Notice that the spin label has been suppressed from \( \mathcal{F} \) since this function is independent of \( \sigma_1 = \sigma_2 = \sigma \). If we would limit ourselves to considering only positive values of the external frequency \( \Omega = v_1 + v_1' \) in Eq. (2.10), the factor \( \mathcal{F} \) on the right-hand side of Eq. (2.9) could be replaced by the corresponding Fourier transform \( L \) of the two-particle correlation function (2.8). The two functions \( \mathcal{F} \) and \( L \), in fact, coincide provided \( v_1 + v_1' > 0 \), as can be seen from the corresponding Lehmann representations. To extend the calculation of the distribution function in terms of the two-particle correlation function to negative values of \( \Omega \), we define further
\[
\delta G_\sigma^*(\mathbf{k}, \omega; Q, \Omega) = \delta G_{10}(\mathbf{k}, \omega; Q, \Omega) \Theta(\Omega)
\]
\[
+ \delta G_{10}^*(\mathbf{k}, \omega; -Q, -\Omega) \Theta(-\Omega)
\]
(2.11)
with the property \( \delta G_\sigma^*(\mathbf{k}, \omega; Q, \Omega) = \delta G_\sigma(\mathbf{k}, \omega; -Q, -\Omega) \).
The phase-space distribution function \( \delta n(\mathbf{k}; Q, \Omega) \) (to first order in the external potential) is then obtained from \( \delta G_\sigma \) in the following way:
\[
\delta n(\mathbf{k}; Q, \Omega) = \delta g(\mathbf{k}, \mathbf{e}_k; Q, \Omega) - 2\delta g(\mathbf{e}_k) \Phi_{\text{ext}}(Q, \Omega).
\]
(2.12)
Here, \( e_k = k^2/2m - \mu \) (\( \mu \) being the chemical potential),
\[
\delta g(\mathbf{k}, \omega; Q, \Omega) = -\sum_\sigma \int \frac{d\mathbf{k}'}{2\pi} \delta G_\sigma(\mathbf{k}|\mathbf{k}', \omega; Q, \Omega)
\]
represents the deviation of the distribution function from local equilibrium, and the term proportional to the Dirac \( \delta \) function \( \delta (e_k) \) is the linear variation of the corresponding local-equilibrium distribution. The above procedure to define the phase-space distribution function \( \delta n \) combines the partitioning of \( \delta n \), as in Eq. (2.12), from the approach of Ref. 17 with the idea of integrating out the magnitude of the wave vector \( \mathbf{k} \), as in Eq. (2.13), from the approach of Ref. 19. The latter has been followed by Hershfield and Ambegaokar in their treatment of the impurity kinetic equation.6

Expression (2.13) can be calculated by the same diagrammatic structure used for the response functions but for a final integration over the direction \( \mathbf{k} \). The two contributions on the right-hand side of Eq. (2.12) can thus be mapped into the ordinary dynamic (++) and static (++ and --) parts of the diagrammatic structure. Details are given in Appendix A.

We next show how the Boltzmann equation results from Eq. (2.12) in the two cases of interest.

III. BOLTZMANN EQUATION IN THE PRESENCE OF NONMAGNETIC IMPURITIES: BACKSCATTERING EFFECT IN THE COLLISION INTEGRAL

The Boltzmann equation for strongly disordered nonmagnetic impurities has already been derived by Hershfield and Ambegaokar.6 In this section we reproduce their result with the alternative methods described in Sec. II and Appendix A, and with the use of somewhat different approximations that will serve in the next section to deal with the magnetic impurity case.

In the presence of nonmagnetic impurities the (quenched) impurity average [which is implied in the
Bethe-Salpeter equation (A1) can be conveniently performed by assuming an on-site impurity potential \( u(\tau) \) with Gaussian distribution \( u(\tau)u(\tau') \)
\[ = \left[ 2\pi N_0(\epsilon_F) \right]^{-1} \delta_r(\tau - \tau') \]
where \( N_0(\epsilon_F) \) is the free-particle density of states per spin component at the Fermi level and the overbar denotes the average over the impurity configurations.\(^{20}\) The expansion parameter
\[ t = \frac{1}{2\pi N_0(\epsilon_F)D_0} \]  
(3.1)
where \( D_0 = 2\tau \mu m^{-1} d^{-1} \) is the Drude diffusion coefficient, then classifies successive approximations in the strength of the impurity potential (loop expansion).
In particular, the ordinary Boltzmann equation with isotropic impurity scattering can be recovered to zeroth order in \( t \) by approximating the irreducible two-particle kernel \( \Xi_k \) of Eq. (A1) by the Born collision contribution depicted in Fig. 1(a). Inclusion of quantum interference to first order in \( t \) requires us, in addition, to consider the crossed ladder contribution [defined in Fig. 1(b)] plus the decorations needed to keep the approximation conserving [Fig. 1(b)]. The relevance of these diagrams to weak localization stems from the fact that the crossed ladder propagator
\[ L_C(k,k';\Omega) = \frac{1}{2\pi N_0(\epsilon_F)^2} \frac{1}{D_0(k+k')^2 - i\Omega} \]  
(3.2)
is strongly peaked in the backscattering direction \( k' = -k \) for small values of \( \Omega \). A Boltzmann equation can then be obtained to first order in \( t \) by combining the methods of Sec. II and Appendix A with a number of approximations introduced by Hershfield and Ambegaokar.\(^6\) Specifically, we could do the following.

(i) Approximate
\[ (-i\Omega + iv_F\hat{k} \cdot Q) \delta g(\hat{k},\omega;Q,\Omega) + i\Omega \Omega \delta(\omega) \Phi_{\text{ext}}(Q,\Omega) \]
\[ = -\frac{1}{\tau} \left[ \delta g(\hat{k},\omega;Q,\Omega) - \int \frac{d\hat{k}}{S_d} \delta g(\hat{k},\omega;Q,\Omega) \right] + \frac{2\tau}{\tau} I_d(\Omega) \left[ \delta g(-\hat{k},\omega;Q,\Omega) - \int \frac{d\hat{k}}{S_d} \delta g(\hat{k},\omega;Q,\Omega) \right] \]  
(3.5)
where \( v_F \) is the Fermi velocity, \( S_d \) stands for the solid angle in \( d \) dimensions, and
\[ I_d(\Omega) = \int \frac{d^d q}{(2\pi)^d q^2 - i\Omega} \]  
(3.6)
is logarithmically singular in two dimensions. (Here and in the following singular integrals, the cutoff frequency \( \Omega \) is understood to be divided by \( D_0 \).) Notice that the factor \( \Omega \delta(\omega) \) in the driving term on the left-hand side of Eq. (3.5) derives from the restriction \(-\Omega/2 < \omega < \Omega/2\), which distinguishes the dynamical contribution.

The linearized kinetic equation for the full distribution function \( \delta n \) can be obtained directly from Eq. (3.5) with the prescription of Sec. II. Specifically, the static contribution of Eq. (2.12), being independent of the direction \( \hat{k} \), has no effect on the right-hand side of Eq. (3.5), while it replaces the driving term on the left-hand side of Eq. (3.5) by
\[ -2\Phi_{\text{ext}}(Q,\Omega)\Phi_{\text{ext}}(\Omega) \]  
(3.7)
where \( 2\Xi E(\omega) \) is identified with the zero-field value \( n_0(\omega) \). One obtains\(^2\)
\[ (-i\Omega + iv_F\hat{k} \cdot Q) \delta n(\hat{k};Q,\Omega) + iv_F\hat{k} \cdot Q \delta \Omega(\xi,\Omega) \Phi_{\text{ext}}(Q,\Omega) \]
\[ = -\frac{1}{\tau} \left[ \delta n(\hat{k};Q,\Omega) - \int \frac{d\hat{k}}{S_d} \delta n(\hat{k};Q,\Omega) \right] + \frac{2\tau}{\tau} I_d(\Omega) \left[ \delta n(-\hat{k};Q,\Omega) - \int \frac{d\hat{k}}{S_d} \delta n(\hat{k};Q,\Omega) \right] \]  
(3.7)
with \( |k| = |k'| \), thus reproducing the result by Hershfield and Ambegaokar in linearized form.

Before discussing Eq. (3.7) further, it is worthwhile to describe how it can be obtained by an alternative strategy that avoids approximations (i)–(iii) above. This is especially important for the magnetic impurity case discussed in the next section, for which these approximations fail to recover the diffusive form of the density-density response function as well as to render the theory renormalizable in the usual way.

We notice at the outset that, consistent with the hydrodynamic regime we are interested in, approximations (i) and (ii) above can be easily disposed of by expanding the single-particle Green’s functions entering the irreducible scattering terms of Fig. 1(b) to order \( \Omega \) and to order \( q'Q^d \) with \( \alpha + \beta \leq 2 \), \( q \) being the one-loop wave vector defined in Eq. (3.6). By the same token, the ansatz (3.4) needs to be replaced by a different prescription to disentangle the irreducible scattering terms from the distribution function itself in the Bethe-Salpeter equation (A1).

This disentangling is, in fact, the crucial step to map the diagrammatic structure onto the collision integral of the Boltzmann equation. To this end, it is sufficient to evaluate this structure up to the required order in the disorder parameter \( t \), namely, to first order in the nonmagnetic impurity case and to second order in the magnetic impurity case. We thus manipulate the terms of order \( t \) corresponding to Fig. 1(b) in the following way.

(i) We replace the function \( L \) on the right-hand side of Eq. (A1) by its free-electron-hole-pair approximation

\[
L^{(0)}[k + Q/2,\omega + \Omega/2; -k + Q/2, -\omega + \Omega/2] = G_+ \left[ k + Q/2, \omega + \Omega/2 \right] G_- \left[ k - Q/2, -\omega - \Omega/2 \right].
\]  

(ii) According to the Hikami procedure, we expand the vertex

\[
\begin{aligned}
J_{++-} &= \frac{Q}{2} - q - \frac{Q}{2} - \frac{Q}{2} + q + \frac{Q}{2} \hat{k}, \Omega \\
J_{++} &= \frac{Q}{2} - q - \frac{Q}{2} - \frac{Q}{2} \hat{k}, \Omega, 1 \int \frac{d\hat{k}}{S_d} J_{++-} \left[ k - \frac{Q}{2}, -q - \frac{Q}{2}, -q + \frac{Q}{2} \hat{k}, \Omega \right] \\
J_{+-} &= \frac{Q}{2} - q - \frac{Q}{2} - \frac{Q}{2} \hat{k}, \Omega, 1 \int \frac{d\hat{k}}{S_d} J_{++-} \left[ -q - \frac{Q}{2}, -q - \frac{Q}{2}, -q + \frac{Q}{2} \hat{k}, \Omega \right].
\end{aligned}
\]  

where the factors

\[
J_{+ \cdots \cdots}(u, \ldots, v, \ldots) \equiv \int \frac{de_k}{2\pi} G_+ \left[ k + u; \omega + \Omega/2 \right] \cdots G_- \left[ k + v; -\omega - \Omega/2 \right] \cdots
\]  

result in integrating Eq. (A1) over \( e_k \), to the required order in the external parameters \( (Q, \Omega) \) and in the one-loop wave vector \( q \). This simplification stems from the fact that in the present theory the dominant role leading to the singular behavior is played by the crossed ladder propagator (3.2). The single-particle Green’s functions in each diagram of Fig. 1(b) thus act effectively as a kind of vertex function depending on wave vector and frequency according to the scheme depicted in Fig. 2.

(iii) We discard a term proportional to \( \Lambda^d \) since it is of no relevance for localization phenomena.

(iv) We manipulate the resulting expression so as to reintroduce the function (3.8).

The contribution of the diagrams of Fig. 1(b) to Eq. (A1) then reads

\[
\int \frac{de_k}{2\pi} G_+ \left[ k + Q/2, \omega + \Omega/2 \right] G_- \left[ k - Q/2, -\omega - \Omega/2 \right] \int \frac{d\hat{k}'}{(2\pi)^d} E_{(1)}^{(b)}(k, k'; Q, \Omega) L^{(0)}[k + Q/2, \omega + \Omega/2; k' + Q/2, -\omega + \Omega/2] = \frac{1}{\tau} - i\Omega + i\nu_{\hat{k}'} \hat{k}' Q \left[ \frac{2\pi}{\tau} \right] \int \frac{de_k}{2\pi} \left[ -k + Q/2, \omega + \Omega/2; k + Q/2, -\omega + \Omega/2 \right] - \frac{d\hat{k}'}{S_d} \int \frac{d\hat{k}''}{2\pi} \left[ -k' + Q/2, \omega + \Omega/2; k' + Q/2, -\omega + \Omega/2 \right].
\]  

Consistent with our earlier approximation, whereby the function \( L \) has been replaced by \( L^{(0)} \), we can at this point proceed in reverse and interpret the function \( L^{(0)} \) in Eq. (3.11) as representing the full \( L \). This replacement, besides being consistent to first order in \( t \), can actually serve to reorganize the whole diagrammatic structure associated with the Bethe-Salpeter equation (A1), by taking advantage of the fact that the Born collision term [Fig. 1(a)] is assumed to be isotropic. Equation (3.5) is then recovered from the result (3.11).
The kinetic equation (3.7) for the full distribution function $\delta n$ can be cast into the general form of a Boltzmann equation for $n = n_0 + \delta n$ (after back Fourier transforming from $Q$ to $R$):\cite{24}

$$\left[-i\Omega + v_F \hat{k} \cdot \nabla_R - \nabla_R \Phi_{\text{ext}}(R, \Omega) \right] n(k; R, \Omega) = -\int \frac{d \hat{k}'}{S_d} W_0(\hat{k}, \hat{k}'; \Omega) \left[ n(k; R, \Omega) - n(k'; R, \Omega) \right],$$

(3.12)

where the mixed space ($R$) and frequency ($\Omega$) representation is kept in order to use $\Omega$ as the infrared cutoff of the theory, according to a standard procedure in localization theory. To first order in $t$, the effective scattering kernel $W_0$ that characterizes the collision integral of Eq. (3.12) is given by

$$W_0(\hat{k}, \hat{k}'; \Omega) = \frac{1}{\tau} \left[ 1 + 2 t I_d(\Omega) \left[ S_d \delta(\hat{k} + \hat{k}') - 1 \right] \right],$$

(3.13)

which exhibits a marked backscattering effect in accordance with the standard interpretation of localization in terms of quantum interference.\cite{3,4} Notice that the term of first order in $t$ does not contribute to the total scattering rate, namely,

$$\int \frac{d \hat{k}'}{S_d} W_0(\hat{k}, \hat{k}'; \Omega) = \frac{1}{\tau},$$

(3.14)

which remains the same as in the Born approximation.

One may then picture the effect of the weak-localization terms on $W_0$ as in Fig. 3, where a small portion (of the order $t$) of the scattering probability is subtracted uniformly over the solid angle $S_d$ and concentrated in a narrow cone about the back direction $\hat{k}' = -\hat{k}$. Notice also that the collision integral of Eq. (3.12) is formally local in the position variable $R$ although the building up of quantum interference actually occurs over a macroscopic portion of the sample. We shall return to this question in the next section dealing with the magnetic impurity case.

One may readily verify that the total density and current are correctly reproduced by the kinetic equation (3.7) to first order in $t$. These and other checks will be discussed in Sec. V in connection with the diffusion equation.

IV. BOLTZMANN EQUATION IN THE PRESENCE OF MAGNETIC IMPURITIES: SPATIAL NONLOCALITY EFFECTS IN THE COLLISION INTEGRAL

In this section we extend the Boltzmann-type description to the localization in the magnetic impurity case. This approach will enable us to interpret the localization mechanism in the absence of time-reversal invariance through the functional form of the effective scattering kernel of the Boltzmann equation, on the same footing of the nonmagnetic impurity case discussed in the previous section.

We adopt in the following the usual assumptions of the theory of localization and consider the (quenched) magnetic impurities to act effectively as a random magnetic field that violates time-reversal invariance. The average over the magnetic impurities can then be performed by assuming an on-site impurity potential $u_m(\tau)$ with Gaussian distribution and random orientations for the magnetic spins, such that

$$u_m(\tau) S I S I' = \frac{1}{2 N} \delta(\tau - \tau') \frac{S(S + 1)}{3} \delta_{ii'},$$

(4.1)

where $i$ and $i'$ label the spin components. In this expression

$$\bar{u}_m^2 S(S + 1) = \frac{1}{2 N} \frac{\delta(\epsilon_F)}{\tau_m},$$

(4.2)

identifies the magnetic scattering time $\tau_m$, which combines with the nonmagnetic contribution to yield the total scattering time $\tau$ entering the single-particle Green's function (A2).\cite{25}
The occurrence of a nontrivial spin structure at each vertex between the electron (hole) lines and the magnetic impurities lines in the diagrammatic structure makes the treatment of weak localization more involved than for the nonmagnetic impurity case considered in Sec. III. In particular, the infrared singularity of the crossed ladder propagator is now cut off by the appearance of masses $\propto \tau_m^{-1}$ in both singlet and triplet particle-particle channels.\footnote{One is thus forced to consider more complex diagrammatic terms such as the crossing of two or more particle-hole direct ladder propagators.\footnote{Specifically, in the presence of magnetic impurities the relevant contributions to the irreducible two-particle kernel $\Xi_s$ of Eq. (A1) consist, up to the second order in the expansion parameter (3.1), of the total Born collision contribution $[2\pi N_0(\varepsilon_F)\tau]^{-1}$ [Fig. 4(a)] and of the diagrams obtained by intersecting two and three direct ladders [Figs. 4(b) and 4(c), respectively] plus the decorations needed to keep the approximation conserving. Taking the electronic spin into proper account to extract the singlet component of the two-particle kernel further requires us to multiply the three ladders diagrams of Fig. 4(c) by an overall factor of 2. Moreover, each direct ladder [defined in Fig. 4(d)] has to be multiplied by a factor $\frac{1}{2}$ which pertains to its singlet component, while its triplet component can be neglected owing to the occurrence of a mass $\propto \tau_m^{-1}$. It is convenient to describe the calculation of the two and three ladders diagrams separately.}

A. Contribution of the two ladders diagrams

The sixteen terms of the integral equation (A1) which contain two crossing direct ladders in their kernel $\Xi_s$ can be organized into two groups characterized by the wave-vector dependence of the function $L$ on the right-hand side. Specifically, in four terms [as in the first term of Fig. 4(b)], $k'$ is replaced by $k+q_2-q_1$, where $q_1$ and $q_2$ are the loop wave vectors that constitute the arguments of the direct ladders.

\[
\int \frac{d\epsilon_k}{2\pi} \mathcal{D}_s(k,\epsilon;Q,\Omega) = (\frac{1}{2})^2 \int \frac{d\mathbf{q}_1}{(2\pi)^d} \int \frac{d\mathbf{q}_2}{(2\pi)^d} \mathcal{L}(\mathbf{q}_1,\Omega) \mathcal{L}(\mathbf{q}_2,\Omega) \times
\]

\[
\left[ a_{+++} + a_{++} \right] \left[\begin{array}{c}
\frac{Q_2}{2} - \frac{Q_1}{2} - \frac{Q_2}{2} - \frac{Q_2}{2} - \frac{Q_2}{2} + \frac{Q_1}{2} + \frac{Q_1}{2} \end{array} \right] + 2I_{++} \left[\begin{array}{c}
-\frac{Q_1}{2} - \frac{Q_1}{2} - \frac{Q_2}{2} - \frac{Q_2}{2} \end{array} \right]
\]

\[
+ I_{++} \left[\begin{array}{c}
-\frac{Q_1}{2} - \frac{Q_1}{2} - \frac{Q_2}{2} + \frac{Q_2}{2} + \frac{Q_2}{2} \end{array} \right] - I_{++} \left[\begin{array}{c}
-\frac{Q_1}{2} - \frac{Q_1}{2} - \frac{Q_2}{2} + \frac{Q_2}{2} - \frac{Q_2}{2} \end{array} \right]
\]

\[
\times \left[\begin{array}{c}
\frac{Q_2}{2} - \frac{Q_1}{2} + \frac{Q_1}{2} + \frac{Q_1}{2} \end{array} \right] \right]
\]

\[
\mathcal{L}(\mathbf{q},\Omega) = \frac{1}{2\pi N_0(\varepsilon_F)\tau^2} \frac{1}{D_0 q^2 - i\Omega}.
\]

A dependence on $\Omega$ results then from these four terms upon integrating Eq. (A1) over $\epsilon_k$ to get the dynamic part of the distribution function. In the remaining twelve terms [as in the second term of Fig. 4(b)], on the other hand, the wave vector $k'$ is fully integrated over. In the following, we shall indicate by $\mathcal{D}_s(k,\epsilon;Q,\Omega)$ (unaveraged) and $\mathcal{D}_s(k,\epsilon;Q,\Omega)$ (averaged) the contributions of the above two groups of terms, in the order.

To proceed further, we adopt the prescriptions discussed in Sec. III and replace the function $\mathcal{L}$ on the right-hand side of Eq. (A1) by its approximation (3.8). In this way, we have to calculate the following angular Hi-kami vertices.\footnote{Specifically, in the presence of magnetic impurities the relevant contributions to the irreducible two-particle kernel $\Xi_s$ of Eq. (A1) consist, up to the second order in the expansion parameter (3.1), of the total Born collision contribution $[2\pi N_0(\varepsilon_F)\tau]^{-1}$ [Fig. 4(a)] and of the diagrams obtained by intersecting two and three direct ladders [Figs. 4(b) and 4(c), respectively] plus the decorations needed to keep the approximation conserving. Taking the electronic spin into proper account to extract the singlet component of the two-particle kernel further requires us to multiply the three ladders diagrams of Fig. 4(c) by an overall factor of 2. Moreover, each direct ladder [defined in Fig. 4(d)] has to be multiplied by a factor $\frac{1}{2}$ which pertains to its singlet component, while its triplet component can be neglected owing to the occurrence of a mass $\propto \tau_m^{-1}$. It is convenient to describe the calculation of the two and three ladders diagrams separately.}
\[
\int \frac{d\epsilon_k}{2\pi} D_n(k, \omega; Q, \Omega) = \left(\frac{1}{2}\right)^2 \int \frac{d\mathbf{q}_1}{(2\pi)^d} \int \frac{d\mathbf{q}_2}{(2\pi)^d} L_D(\mathbf{q}_1, \Omega) L_D(\mathbf{q}_2, \Omega)
\]
\[
\times \frac{1}{\tau} \int \frac{d\mathbf{k}'}{S_d} \left[ \mathcal{J}_{++-} \left[ \frac{Q}{2}, \mathbf{q}_1 - \frac{Q}{2}, \frac{Q}{2}, \frac{Q}{2}, -\frac{Q}{2}, -\frac{Q}{2}, -\frac{Q}{2}, \mathbf{k}', \Omega \right] \right.
\]
\[
+ \mathcal{J}_{+-} \left[ \frac{Q}{2}, -\frac{Q}{2}, -\frac{Q}{2}, -\frac{Q}{2}, \mathbf{k}', \Omega \right]
\]
\[
\times I_{+++} \left[ \frac{Q}{2}, \mathbf{q}_2 - \frac{Q}{2}, \mathbf{q}_2 - \frac{Q}{2}, \frac{Q}{2}, -\frac{Q}{2}, -\frac{Q}{2}, -\frac{Q}{2}, \mathbf{k}', \Omega \right] \]
\[
+ \mathcal{J}_{++-} \left[ \frac{Q}{2}, \mathbf{q}_2 - \frac{Q}{2}, \frac{Q}{2}, -\frac{Q}{2}, -\frac{Q}{2}, -\frac{Q}{2}, \mathbf{k}', \Omega \right]
\]
\[
\times I_{+++} \left[ \frac{Q}{2}, \mathbf{q}_2 - \frac{Q}{2}, \mathbf{q}_2 - \frac{Q}{2}, \frac{Q}{2}, -\frac{Q}{2}, -\frac{Q}{2}, -\frac{Q}{2}, \mathbf{k}', \Omega \right] \]
\[
+ \mathcal{J}_{++-} \left[ \frac{Q}{2}, \mathbf{q}_2 - \frac{Q}{2}, -\frac{Q}{2}, -\frac{Q}{2}, -\frac{Q}{2}, -\frac{Q}{2}, \mathbf{k}', \Omega \right]
\]
\[
\times I_{+++} \left[ \frac{Q}{2}, \mathbf{q}_2 - \frac{Q}{2}, -\frac{Q}{2}, -\frac{Q}{2}, -\frac{Q}{2}, -\frac{Q}{2}, \mathbf{k}', \Omega \right] \right] ,
\]
\[(4.5)\]

with the additional notation
\[
I_{+++} \ldots \ldots (u, \ldots, v, \ldots; \Omega) \equiv \frac{1}{\tau} \int \frac{d\mathbf{k}'}{S_d} \mathcal{J}_{+++} \ldots \ldots (u, \ldots, v, \ldots; \mathbf{k}', \Omega) .
\]
\[(4.6)\]

Notice that the dependence on the direction \(\mathbf{k}'\) has been explicitly kept in Eq. (4.5) although the notation (4.6) could have been used to represent the corresponding averaging over \(\mathbf{k}'\). This is because the occurrence of an explicit dependence on \(\mathbf{k}'\) will be needed to properly identify the effective scattering kernel of the Boltzmann equation. Notice also that in both Eqs. (4.4) and (4.5) the integrals over \(\mathbf{q}_1\) and \(\mathbf{q}_2\) are understood to be cut off in the ultraviolet.

Calculation of Eqs. (4.4) and (4.5) proceeds now by expanding the associated angular Hikami vertices to order \(\Omega\) and to order \(q_i Q^i (i = 1, 2)\), with \(\alpha + \beta \leq 2\). Upon reintroducing the function (3.8) we obtain
\[
\int \frac{d\epsilon_k}{2\pi} D_n(k, \omega; Q, \Omega) = \left[ \frac{1}{\tau} - i \Omega + i \omega \mathbf{k} \cdot \mathbf{Q} \right]^{-1} I_{\mathbf{k}^2}(\Omega) \left[ 1 - 8\tau_i \Omega_\tau + 2\tau \sqrt{\mathbf{k} \cdot \mathbf{Q} - 2\tau^2 v_J^2 (\mathbf{k} \cdot \mathbf{Q})^2} \right]
\]
\[
\times \int \frac{d\epsilon_k}{2\pi} L^{(0)} \left[ \frac{k_2 + Q_2}{2}, \omega + Q_2; -k_2 + Q_2, -\omega + \Omega_2 \right] ,
\]
\[(4.7)\]
and
\[
\int \frac{d\mathbf{q}_k}{2\pi} D_\sigma (\mathbf{k}, \omega; Q, \Omega) = - \left[ \frac{1}{\tau - i\Omega + i\nu_F \mathbf{k} \cdot Q} \right]^{-1} \times \frac{t^2}{2\tau} I_d^2(\Omega) \int \frac{d\mathbf{k}'}{S_d} \left[ \frac{1}{\tau - 7i\Omega \tau + 3\tau D_0 Q^2 + 2i\tau \nu_F (\mathbf{k} + \mathbf{k}') \cdot Q - 2\tau^2 \nu_F^2 (\mathbf{k} \cdot Q)^2 + (\mathbf{k}' \cdot Q)^2} \right] \times \int \frac{d\mathbf{q}'}{2\pi} \mathcal{L}^{(0)} \left[ \frac{Q}{2}, -\omega = \frac{\Omega}{2}; -\mathbf{k}' = \frac{Q}{2}, -\omega = \frac{\Omega}{2} \right],
\]
(4.8)
where all polynomial expressions are understood to be kept up to the required order in the external variables. Notice that in both equations a term proportional to $\Lambda^6 \ln(\Omega / D_0 \Lambda^2)$ has been discarded by the same line of arguments given in Sec. III.23 Akin to Eq. (3.11) for the nonmagnetic impurity case, the factor $[(1/\tau) - i\Omega + i\nu_F \mathbf{k} \cdot Q]^{-1}$ has been evidenced in Eqs. (4.7) and (4.8) to promptly identify the driving term of the Boltzmann equation. In addition, the function $\mathcal{L}^{(0)}$ can eventually be replaced by the full $\mathcal{L}$.

B. Contribution of the three ladders diagrams

In all nine terms of the integral equation (A1) which contain three crossing direct ladders in their kernel $\Xi_\sigma$ [cf. Fig. 4(c)], the wave vector $\mathbf{k}'$ of the function $\mathcal{L}$ on the right-hand side of Eq. (A1) is fully integrated over. If we then replace this function by its approximation (3.8), we can represent the contribution of the three ladders diagrams in the following form:
\[
\int \frac{d\mathbf{q}_k}{2\pi} T(\mathbf{k}, \omega; Q, \Omega) = \frac{2}{\pi^3} \int \frac{d\mathbf{q}^1_1}{(2\pi)^d} \int \frac{d\mathbf{q}^2_2}{(2\pi)^d} L_D(\mathbf{q}^1_1, \Omega) L_D(\mathbf{q}^2_2, \Omega) \times L_D(\mathbf{q}^1_1 + \mathbf{q}^2_2 - \mathbf{Q}, \Omega) \nu(\mathbf{q}^1_1, \mathbf{q}^2_2, \mathbf{Q}; \mathbf{k}', \Omega) 2\pi N_0(\nu_F) \int \frac{d\mathbf{k}'}{S_d} \mathcal{L}(\mathbf{q}^1_1, \mathbf{q}^2_2, \mathbf{Q}; \mathbf{k}', \Omega),
\]
(4.9)
where
\[
\nu(\mathbf{q}^1_1, \mathbf{q}^2_2, \mathbf{Q}; \mathbf{k}', \Omega) = \mathcal{S}_{++} \left[ \frac{\mathbf{q}^1_1 - \mathbf{Q}}{2}, -\mathbf{Q} \right] - \mathcal{S}_{++} \left[ \frac{\mathbf{q}^2_2 - \mathbf{Q}}{2}, -\mathbf{Q} \right] \mathbf{k}', \Omega \right] + \mathcal{S}_{++} \left[ \frac{\mathbf{q}^1_1 - \mathbf{Q}}{2}, -\mathbf{Q} \right] \mathbf{k}', \Omega \right] + \mathcal{S}_{++} \left[ \frac{\mathbf{q}^2_2 - \mathbf{Q}}{2}, -\mathbf{Q} \right] \mathbf{k}', \Omega \right]
\]
(4.10)
is an additional angular Hikami vertex.

A characteristic feature of the three ladders contribution (4.9) is the appearance in the middle ladder of a dependence on the external momentum $\mathbf{Q}$. This dependence cannot be dropped altogether since it contributes leading terms to the scattering kernel. To extract these contributions we expand the middle ladder up to second order in $\mathbf{Q}$ while keeping $\Omega$ finite. As we have already remarked after Eq. (3.12), this procedure amounts to taking the external frequency $\Omega$ as the infrared cutoff of the theory, thereby exploring the restricted region $D_0 Q^2 \ll \Omega$ with the provision that the final result will be analytically continued in the whole hydrodynamic region where $D_0 Q^2$ and $\Omega$ are comparable.

The vertex (4.10) can then be expanded to order $\Omega$ and to order $q_i^\alpha Q^\beta$ ($i = 1, 2$), with $\alpha + \beta \leq 2$, in analogy with the prescription used to get the two ladders contributions (4.7) and (4.8). This procedure turns out to be sufficient even in the presence of three singular denominators due to the direct ladders (4.3), because the relevant expansion
\[
\nu(\mathbf{q}^1_1, \mathbf{q}^2_2, \mathbf{Q}; \mathbf{k}', \Omega) = \tau^2 [2\tau D_0 Q^2 - 2i\Omega \tau + 4\tau D_0 (\mathbf{q}^1_1 + \mathbf{q}^2_2)] + 6\tau D_0 (\mathbf{q}^1_1 + \mathbf{q}^2_2) + 5\tau D_0 (\mathbf{q}^1_1 + \mathbf{q}^2_2) - 3\tau^2 \nu_F^2 (\mathbf{k} \cdot \mathbf{Q})^2 (\mathbf{q}^1_1 + \mathbf{q}^2_2) - 3\tau^2 \nu_F^2 (\mathbf{k} \cdot \mathbf{Q})^2 (\mathbf{q}^1_1 + \mathbf{q}^2_2) - 3\tau^2 \nu_F^2 (\mathbf{k} \cdot \mathbf{Q})^2 (\mathbf{q}^1_1 + \mathbf{q}^2_2) - 3\tau^2 \nu_F^2 (\mathbf{k} \cdot \mathbf{Q})^2 (\mathbf{q}^1_1 + \mathbf{q}^2_2)
\]
(4.11)
do not contain a constant term. When multiplying together the two vertices of Eq. (4.9), however, it is necessary to keep all terms to order $\mathbf{q}^\alpha$ and to order $\mathbf{q}^\alpha \mathbf{Q}^\beta$ (as well as $\mathbf{q}^\alpha \mathbf{Q}^\beta$) for the purpose of extracting the leading contributions to the scattering kernel of the Boltzmann equation. In the following we shall consistently retain only the minimal number of terms that satisfy this requirement, since we are interested in the effective scattering processes leading to the critical behavior.

To proceed further in the calculation of Eq. (4.9), it is convenient to perform at the outset certain averages over the
angular variables of both vectors $q_1$ and $q_2$ in $2d$ dimensions. These averages generalize to many pairs of scalar products the well-known result
\[
\langle \langle q \cdot a | q \cdot b \rangle \rangle = \frac{\langle a \cdot b \rangle}{d} \langle \langle q^2 \rangle \rangle ,
\]
(4.12)
where $\langle \langle \cdots \rangle \rangle$ stands for the average over the angular variables of the vector $q$. Equation (4.12) has already been used to get the two ladders contributions (4.7) and (4.8). For completeness we quote the relevant formulas in Appendix B.

After a long but straightforward calculation, we get eventually for the three ladders contribution
\[
\int \frac{dk}{2\pi} T(k, \omega; Q, \Omega) = \left[ \frac{1}{\tau} - i \Omega + i v_f \hat{k} \cdot Q \right]^{-1} \times \frac{i^2}{4\sigma} \int \frac{d\hat{k}}{S_d} \left[ J_1(\Omega) + (1 - 2i \Omega \tau) \gamma_1(\Omega; \hat{k} \cdot \hat{k}') + i \tau v_f (\hat{k} + \hat{k}') \cdot Q \gamma_3(\Omega; \hat{k} \cdot \hat{k}') \right.
\]
\[+ \tau \gamma_0(\hat{k} \cdot Q; \hat{k} \cdot Q) \gamma_3(\Omega; \hat{k} \cdot \hat{k}')
\]
\[+ \tau \gamma_0(\hat{k} \cdot Q; \hat{k} \cdot \hat{k}') \gamma_3(\Omega; \hat{k} \cdot \hat{k}') + \tau \gamma_2(\Omega; \hat{k} \cdot \hat{k}') \right]
\]
\[\times \int \frac{d\varepsilon_k}{2\pi} L(0) \left[ k' + \frac{Q}{2}, \omega + \frac{\Omega}{2}; -k' + \frac{Q}{2}, -\omega + \frac{\Omega}{2} \right] ,
\]
(4.13)
where $\gamma_i(\Omega; \hat{k} \cdot \hat{k}')$ ($i = 1, 2, \ldots, 5$) are polynomials up to second order in $\hat{k} \cdot \hat{k}'$ with coefficients that depend on the two-loop integrals $J_i(\Omega)$ ($i = 1, 2, \ldots, 13$). The expressions of $\gamma_i(\Omega; \hat{k} \cdot \hat{k}')$ and $J_i(\Omega)$ are listed in Appendix B, where we also show how the relevant singular part of the two-loop integrals $J_i(\Omega)$ can be extracted. Notice that in deriving Eq. (4.13) we have anticipated that the relevant part of $J_i(\Omega)$ is linear in the external frequency $\Omega$.

Not every term in the expression within curly brackets on the right-hand side of Eq. (4.13) is required to obtain the effective scattering kernel of the Boltzmann equation in the hydrodynamic regime. To check this point, it is necessary to combine the three ladders contribution (4.13) with the two ladders contributions (4.7) and (4.8).

C. Effective scattering kernel of the Boltzmann equation

The two and three ladders contributions calculated above can be combined in the form
\[
\int \frac{d\varepsilon_k}{2\pi} \left[ D_2(\varepsilon_k, \omega; Q, \Omega) + D_3(\varepsilon_k, \omega; Q, \Omega) + T(\varepsilon_k, \omega; Q, \Omega) \right]
\]
\[= \int \frac{d\varepsilon_k}{2\pi} \left[ D_2(\varepsilon_k, \omega; Q, \Omega) + D_3(\varepsilon_k, \omega; Q, \Omega) + T(\varepsilon_k, \omega; Q, \Omega) \right]
\]
\[\times \int \frac{d\varepsilon_k}{2\pi} L \left[ k + \frac{Q}{2}, \omega + \frac{\Omega}{2}; -k + \frac{Q}{2}, -\omega + \frac{\Omega}{2} \right]
\]
\[+ V(\hat{k}, \hat{k}'; Q, \Omega) \int \frac{d\varepsilon_k}{2\pi} L \left[ k' + \frac{Q}{2}, \omega + \frac{\Omega}{2}; -k' + \frac{Q}{2}, -\omega + \frac{\Omega}{2} \right] .
\]
(4.14)

Here,
\[
\int \frac{d\varepsilon_k}{S_d} U(\hat{k}, \hat{k}'; Q, \Omega) = \sqrt{\frac{2}{\pi}} \int_{\beta} [1 - 8i \Omega \tau + 2i v_f \hat{k} \cdot Q - 2\tau v_f^2 (\hat{k} \cdot Q)^2]
\]
(4.15)
results from Eq. (4.7), while $V(\hat{k}, \hat{k}'; Q, \Omega)$ is read from Eqs. (4.8) and (4.13). The kernel $V(\hat{k}, \hat{k}'; Q, \Omega)$ turns out to be symmetric under the interchange $\hat{k} \leftrightarrow \hat{k}'$. In addition, from the results of Appendix C one can verify that
\[
\int \frac{d\varepsilon_k}{S_d} V(\hat{k}, \hat{k}'; Q, \Omega) = -\int \frac{d\varepsilon_k}{S_d} U(\hat{k}, \hat{k}'; Q, \Omega) ,
\]
(4.16)
which is obtained after extensive cancellations have occurred (at the leading order of the $e = d - 2$ expansion we are considering). The property (4.16) is important for obtaining the Boltzmann equation, since it enables us to rewrite the right-hand side of Eq. (4.14) in the form of a collision integral, namely, in the form
\[
I(\hat{k}, Q, \Omega) = \int \frac{d\varepsilon_k}{S_d} V(\hat{k}, \hat{k}'; Q, \Omega) \int \frac{d\varepsilon_k}{2\pi} L \left[ k + \frac{Q}{2}, \omega + \frac{\Omega}{2}; -k + \frac{Q}{2}, -\omega + \frac{\Omega}{2} \right]
\]
\[+ \int \frac{d\varepsilon_k}{S_d} V(\hat{k}, \hat{k}'; Q, \Omega) \int \frac{d\varepsilon_k}{2\pi} L \left[ k' + \frac{Q}{2}, \omega + \frac{\Omega}{2}; -k' + \frac{Q}{2}, -\omega + \frac{\Omega}{2} \right] .
\]
(4.17)
Owing to the symmetry of the kernel $V(\hat{\mathbf{k}}, \hat{\mathbf{k}}'; \mathbf{Q}, \Omega)$, $J(\hat{\mathbf{k}}; \mathbf{Q}, \Omega)$ satisfies the condition
\[
\int \frac{d\hat{\mathbf{k}}}{S_d} J(\hat{\mathbf{k}}; \mathbf{Q}, \Omega) = 0.
\] (4.18)

Equation (4.18) guarantees, in particular, that the continuity equation is satisfied. The property (4.16) thus represents a strong check on the procedure we have followed to evaluate the diagrammatic contributions at two-loop order.

The result (4.16) enables us also to simplify considerably the expression of $V(\hat{\mathbf{k}}, \hat{\mathbf{k}}'; \mathbf{Q}, \Omega)$ in the hydrodynamic regime. In fact, since the difference within square brackets on the right-hand side of Eq. (4.17) is at least of order $|\mathbf{Q}|$, it is sufficient to retain only terms up to order $|\mathbf{Q}|$ in the expression of $V(\hat{\mathbf{k}}, \hat{\mathbf{k}}'; \mathbf{Q}, \Omega)$. We then take
\[
V(\hat{\mathbf{k}}, \hat{\mathbf{k}}'; \mathbf{Q}, \Omega) = -\frac{l^2}{2\tau} I^2_d(\Omega) \left[ 1 + 2(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') + i(\hat{\mathbf{k}} + \hat{\mathbf{k}}') \cdot \mathbf{Q} \left[ 1 + \frac{\epsilon}{8} + 2(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}')(1 + \frac{1}{8}\epsilon) \right] \right]
\] (4.19)

to order $\epsilon$ in $\epsilon$ expansion, where $l = \tau v_F$ is the mean free path within the Born approximation. Notice that no frequency renormalization occurs correctly in Eq. (4.19).

The Boltzmann equation in the presence of strongly disordered magnetic impurities is readily derived at this point by adding the Born collision contribution of Fig. 4(a) to expression (4.19). For the dynamic part of the distribution function we obtain [cf. the corresponding Eq. (3.5) for the nonmagnetic impurity case]
\[
(-i\Omega + i\nu_F \hat{\mathbf{k}} \cdot \mathbf{Q}) \delta g(\hat{\mathbf{k}}, \omega; \mathbf{Q}, \Omega) + i\Omega \Phi(\mathbf{Q}, \Omega) = -\int \frac{d\hat{\mathbf{k}}'}{S_d} W(\hat{\mathbf{k}}, \hat{\mathbf{k}}'; \mathbf{Q}, \Omega) \left[ \delta g(\hat{\mathbf{k}}, \omega; \mathbf{Q}, \Omega) - \delta g(\hat{\mathbf{k}}', \omega; \mathbf{Q}, \Omega) \right],
\] (4.20)

where
\[
W(\hat{\mathbf{k}}, \hat{\mathbf{k}}'; \mathbf{Q}, \Omega) = \frac{1}{\tau} + V(\hat{\mathbf{k}}, \hat{\mathbf{k}}'; \mathbf{Q}, \Omega)
\] (4.21)
is the total effective scattering kernel. The kinetic equation for the full distribution function (2.12) follows then directly from Eq. (4.20) by substituting $bn$ for $\delta g$ and replacing $\Omega \Phi(\omega)$ in the driving term with $\nu_F \hat{\mathbf{k}} \cdot \mathbf{Q} \delta(\epsilon_k)$. The resulting equation can, in turn, be cast into the general form of a Boltzmann equation by back Fourier transforming from $\mathbf{Q}$ to $\mathbf{R}$. The occurrence of the $\mathbf{Q}$ dependence of the scattering kernel (4.19), however, requires us to introduce an additional integration over the space variable in the collision integral, as compared with its counterpart in the presence of nonmagnetic impurities on the right-hand side of Eq. (3.12). That is, in the presence of magnetic impurities the Boltzmann equation reads
\[
[-i\Omega + \nu_F \hat{\mathbf{k}} \cdot \mathbf{R} - \mathbf{v}_R \Phi_{ext}(\mathbf{R}, \Omega) - \mathbf{v}_k] n(\mathbf{k}; \mathbf{R}, \Omega) = -\int \frac{d\hat{\mathbf{k}}'}{S_d} \int d\mathbf{R}' W(\hat{\mathbf{k}}, \hat{\mathbf{k}}'; \mathbf{R} - \mathbf{R}', \Omega) \left[ n(\mathbf{k}; \mathbf{R}', \Omega) - n(\mathbf{k}'; \mathbf{R}', \Omega) \right]
\] (4.22)

with $|\mathbf{k}| = |\mathbf{k}'|$. This equation is consistent with the form (4.19) of the scattering kernel if we assume that $W(\hat{\mathbf{k}}, \hat{\mathbf{k}}'; \mathbf{R} - \mathbf{R}', \Omega)$ is sharply peaked about $\mathbf{R}' = \mathbf{R}$ on the scale of the slow (hydrodynamic) variation of the distribution function. The difference $n(\mathbf{k}; \mathbf{R}', \Omega) - n(\mathbf{k}'; \mathbf{R}', \Omega)$ on the right-hand side of Eq. (4.22) can thus be expanded in powers of $\mathbf{R}' - \mathbf{R}'$ up to the first order, and the spatial dependence of the effective scattering kernel can be characterized by its first two moments
\[
W_0(\hat{\mathbf{k}}, \hat{\mathbf{k}}'; \Omega) = \int d\mathbf{R}' W(\hat{\mathbf{k}}, \hat{\mathbf{k}}'; \mathbf{R} - \mathbf{R}', \Omega),
\] (4.23a)
\[
W_1(\hat{\mathbf{k}}, \hat{\mathbf{k}}'; \Omega) = \int d\mathbf{R}' W(\hat{\mathbf{k}}, \hat{\mathbf{k}}'; \mathbf{R} - \mathbf{R}', \Omega) (\mathbf{R}' - \mathbf{R}).
\] (4.23b)

In the magnetic impurity case we have found [cf. Eqs. (4.19) and (4.21)]
\[
W_0(\hat{\mathbf{k}}, \hat{\mathbf{k}}'; \Omega) = \frac{1}{\tau} \left[ 1 - \frac{l^2}{2} J^2_d(\Omega) [1 + 2(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}')] \right],
\] (4.24a)
\[
W_1(\hat{\mathbf{k}}, \hat{\mathbf{k}}'; \Omega) = -\frac{l^2}{2\tau} J^2_d(\Omega) \left[ 1 + \frac{\epsilon}{8} + 2(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}')(1 + \frac{1}{8}\epsilon) \right] l(\hat{\mathbf{k}} + \hat{\mathbf{k}}').
\] (4.24b)

Notice the following features of Eqs. (4.24).

(i) The first moment $W_1$ is directed along the direction $\hat{\mathbf{k}} + \hat{\mathbf{k}}'$, which is in fact the only nonvanishing symmetric combination of the unit vectors $\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}'$ (the other possible symmetric combination $\hat{\mathbf{k}} \times \hat{\mathbf{k}}' + \hat{\mathbf{k}}' \times \hat{\mathbf{k}}$ vanishes identically). This is because the averaging over the impurity configurations restores the symmetry of the effective scattering kernel under the interchange $\hat{\mathbf{k}} \leftrightarrow \hat{\mathbf{k}}'$, which
would otherwise hold only for single impurity scattering events.\textsuperscript{1,11}

(ii) Equations (4.24) can be interpreted with the corresponding expressions for the nonmagnetic impurity case, where $W_0$ is given by Eq. (3.13) while $W_1$ vanishes identically. $W_1$ cannot in fact survive when time-reversal symmetry requires in addition the scattering kernel to be symmetric under the interchange $\mathbf{k} \leftrightarrow -\mathbf{k}'$. This vanishing of $W_1$ is also implicit, although for a different reason, in the (isotropic) Born approximation

$$W_0(\mathbf{k}, \mathbf{k}'; \mathbf{R} - \mathbf{R}', \Omega) = \frac{1}{\tau} \delta(\mathbf{R} - \mathbf{R}')$$

(4.25)

that disregards quantum-interference effects altogether. In this case the presence of the spatial $\delta$ function arises from the classical approximation of considering each scattering process by a short-range potential as an independent event.\textsuperscript{1,11}

(iii) Equations (4.24) show that the breaking of time-reversal symmetry due to the presence of magnetic impurities induces a nonlocality of the effective scattering kernel $W(\mathbf{k}, \mathbf{k}'; \mathbf{R} - \mathbf{R}', \Omega)$, which couples scattering processes occurring in the neighborhood of a given $\mathbf{R}$ within a range of the order of mean free path:

$$\frac{W_1}{W_0} = i(\mathbf{k} + \mathbf{k}')[1 + O(\epsilon)]$$

(4.26)

Moreover, in contrast to the result (3.13) for nonmagnetic impurities, in the presence of magnetic impurities $W_0$ turns out to be only weakly enhanced in the back direction $\mathbf{k}' = -\mathbf{k}$, while $W_1$ vanishes in that direction. In the presence of magnetic impurities the effective reflection of the incident wave then occurs over a broad region of phase space, the finite range (4.26) of the effective scattering kernel representing the relevant distance over which alternative interference patterns result in a coherent wavefront. This remark implies (as we have mentioned at the end of Sec. III) that the quantum-interference effects giving rise to the effective scattering kernels (3.13) and (4.24) are actually built up over macroscopic portions of the sample, after a large number of microscopic scattering events have taken place.

The Boltzmann equation (4.22), as well as its counterpart (3.12) in the presence of nonmagnetic impurities only, shows that alternative mesoscopic scattering channels are associated with different relaxation times. In order to obtain a limiting macroscopic relaxation process (on the hydrodynamic scale) one has then to show how a well-defined diffusion process results from these equations.

V. RECOVERING THE DIFFUSION EQUATION FROM THE BOLTZMANN EQUATION

The appearance of a nontrivial dependence on the external frequency $\Omega$ in the scattering kernel of the Boltzmann transport equations for impurity scattering that we have derived in Secs. III and IV makes somewhat questionable the conventional argument that describes the overall decay of the system toward complete equilibrium through the Boltzmann's H theorem.\textsuperscript{26} If one is interested specifically in determining the slow (hydrodynamic) behavior of the system, however, it is sufficient to single out at the outset from the transport equation the modes which exhibit the slow behavior instead of following the succession of scattering events. To this end, one has to average the distribution function over the allowed region of phase space, thereby performing a sort of coarse-graining procedure on the dynamical behavior of the system. In the present context this corresponds to averaging with equal weight over all possible directions $\hat{k}$ on the Fermi surface, and thus to determining the kinetic equation for the average distribution function

$$\delta n(\mathbf{k}; \Omega) = \int \frac{d\mathbf{k}}{S_d} \delta n(\mathbf{k}; \mathbf{Q}, \Omega) .$$

(5.1)

Equation (5.1) can be interpreted as the (variation of the) energy distribution function.

We carry out the averaging for the nonmagnetic and the magnetic impurity cases separately.

A. Nonmagnetic impurities

In this case one can solve directly the coupled system of equations for $\delta n(\mathbf{k}; \mathbf{Q}, \Omega)$ and $\delta n(-\mathbf{k}; \mathbf{Q}, \Omega)$ that results from the transport equation (3.7) by letting $\mathbf{k} \rightarrow -\mathbf{k}$, and obtain an equation for $[\delta n(\mathbf{k}; \mathbf{Q}, \Omega) + \delta n(-\mathbf{k}; \mathbf{Q}, \Omega)]/2$ that can be readily averaged over $\mathbf{k}$. Under the hydrodynamic conditions $\Omega\tau << 1$ and $D_0Q^2\tau << 1$ one gets

$$(-\Omega + DQ^2)\delta \overline{n}(\mathbf{k}; \mathbf{Q}, \Omega) = -2\delta(\epsilon_k)DQ^2\Phi_{\text{ext}}(\mathbf{Q}, \Omega),$$

(5.2)

where

$$D = D_0[1 - 2iU_d(\Omega)] \rightarrow D_0 \left[ 1 + \frac{f}{2\pi} \ln(\Omega\tau) \right]$$

(5.3)

is the well-known result for the renormalized diffusion coefficient (to leading order in $\epsilon$ expansion).\textsuperscript{3} The last line of Eq. (5.3) follows either directly from Eq. (3.6) by setting $d = 2$ and interpreting $D_0\Lambda^2$ as the inverse of the microscopic scattering time $\tau$, or from Eq. (C13) within dimensional regularization after the pole for $\epsilon \rightarrow 0^+$ has been minimally subtracted.\textsuperscript{27} Notice that at the one-loop order renormalization could have been performed directly on the effective scattering kernel (3.13) of the Boltzmann equation, although it is physically more meaningful to renormalize the diffusion coefficient which describes the behavior of the system over macroscopic time and space scales. Notice also that the diffusion equation (5.2) can formally be mapped into a collisionless Boltzmann equation by replacing the diffusive factor $DQ^2$ with its ballistic counterpart $i\epsilon_d\mathbf{k} \cdot \mathbf{Q}$.

The induced particle density [cf. Eq. (2.1)]

$$\delta n(\mathbf{Q}, \Omega) = N_0(\epsilon_F) \int d\epsilon_k \delta \overline{n}(\mathbf{k}; \mathbf{Q}, \Omega) .$$

(5.4)

can be obtained directly from Eq. (5.2), thereby recovering the diffusive form of the density-density correlation function
\[ \chi_{nn}(Q, \Omega) = \chi_{nn}^{\mu} \frac{DQ^2}{DQ^2 - i\Omega} , \]  
(5.5)

where \( \chi_{nn}^{\mu} = -2N_0(\epsilon_F) \) is the appropriate static limit for noninteracting electrons. The induced particle current [cf. Eq. (2.2)] can instead be obtained by multiplying both sides of Eq. (3.7) by \( \mathbf{k}/m \) and integrating over \( \mathbf{k} \). To leading order in \( Q \) and \( \Omega \) one gets

\[ j(Q, \Omega) = -iQ\Phi_{ext}(Q, \Omega)2N_0(\epsilon_F)D . \]  
(5.6)

Alternatively, one may resort to the continuity equation that results upon averaging over \( \mathbf{k} \) the Boltzmann equation (3.7), owing to the vanishing of the spherical average of the collision integral on its right-hand side.

### B. Magnetic impurities

The occurrence of a nontrivial dependence of the effective scattering kernel (4.19) on the incoming \( \mathbf{k} \) and outgoing \( \mathbf{k}' \) directions renders the angular averaging of the kinetic equation (4.20) in the presence of magnetic impurities more involved. To this end, it is convenient to rewrite first the collision integral on the right-hand side of Eq. (4.20) in the form

\[ \begin{align*}
- \int d\hat{k}' \left[ W(\hat{k}, \hat{k}'; Q, \Omega) [\delta g(\hat{k}, \omega; Q, \Omega) - \delta g(\hat{k}', \omega; Q, \Omega)] \right] \\
= - \frac{1}{\tau_{eff}(\Omega)} [\delta g(\hat{k}, \omega; Q, \Omega) - \delta g(\omega; Q, \Omega)] + [1 - \alpha(\Omega)]i\nu_F \cdot \mathbf{Q} \delta g(\hat{k}, \omega; Q, \Omega) \\
+ [\beta(\Omega)i\nu_F \cdot \mathbf{Q} + \gamma(\Omega)i\nu_F \cdot \tau(\hat{k} - \mathbf{Q})^2 + \delta(\Omega)D_0Q^2] \delta g(\omega; Q, \Omega) ,
\end{align*} \]  
(5.7)

where

\[ \delta g(\omega; Q, \Omega) = \int d\hat{k} g(\hat{k}, \omega; Q, \Omega) \]  
(5.8)

is the average of the dynamic part of the distribution function, and where we have introduced the notation

\[ \alpha(\Omega) = 1 - t^2I_3^{\beta}(\Omega), \quad \beta(\Omega) = \frac{1 - d}{d} t^2I_3^{\beta}(\Omega) , \]

\[ \gamma(\Omega) = \frac{(4 - 10d)}{4d(d + 2)} I_2^{\beta}(\Omega), \quad \delta(\Omega) = - \frac{(2d^2 - d + 2)}{2d(d + 2)} I_2^{\beta}(\Omega) , \]  
(5.9)

\[ \frac{1}{\tau_{eff}(\Omega)} = \frac{1}{\tau} \left[ 1 - \frac{t^2}{2} I_2^{\beta}(\Omega) \right] . \]

Equation (5.7) results from the same procedure we have consistently followed in Secs. III and IV, namely, by replacing \( \delta g \) with its lowest-order expression [cf. Eq. (3.8)]

\[ \delta g(\hat{k}', \omega; Q, \Omega) = \frac{2i\Omega\tau\delta(\omega)}{1 - i\Omega\tau + i\nu_F R \cdot \mathbf{Q}} \Phi_{ext}(Q, \Omega) \]  
(5.10)

whenever required to perform the integration over \( \mathbf{k}' \) explicitly, with the provision that the full \( \delta g(\omega; Q, \Omega) \) is eventually reintroduced in the final expression.

In this way Eq. (4.20) can be readily averaged over \( \mathbf{k} \), to obtain

\[ (-i\Omega + DQ^2)\delta g(\omega; Q, \Omega) = -28i\omega(\Omega)\Phi_{ext}(Q, \Omega) , \]  
(5.11)

where now the renormalized diffusion coefficient is given by

\[ D = D_0 \left[ \frac{\tau_{eff}(\Omega)}{\tau} \right] \left[ \frac{\alpha(\Omega) - \beta(\Omega)}{\gamma(\Omega) - \delta(\Omega)} \right] \]  
(5.12)

Notice that the presence of the factor \( d - 2 = \epsilon \) in front of \( I_2^{\beta}(\Omega) \) in Eq. (5.12) is essential to allow for a renormalization of that expression. Recalling, in fact, that within dimensional regularization \( I_2^{\beta}(\Omega) \) is given by Eq. (C13), Eq. (5.12) reads near two dimensions:

\[ \frac{D}{D_0} = 1 + \frac{\epsilon}{4} \frac{t^2}{(2\pi)^2} \]  
(5.13)

\[ = 1 + \frac{t^2}{16\pi^2} \frac{[1 + \epsilon \ln(\Omega\tau) + \cdots]}{\epsilon} \]

\[ \rightarrow 1 + \frac{t^2}{16\pi^2} \ln(\Omega\tau) , \]

after the pole for \( \epsilon \rightarrow 0^- \) (with an \( \Omega \)-independent residue) has been minimally subtracted, thereby recovering the standard scaling result. For this reason and in contrast to what we have found at one-loop order, at two-loop order renormalization cannot be performed directly on the effective scattering kernel (4.19) of the Boltzmann equation.

The diffusion equation of the form (5.2) for the average \( \delta n \) of the full distribution function can be obtained from Eq. (5.11) upon replacing \( \delta g \) therein through the angular average of Eq. (2.12). Similarly, one can show that the induced current still has the form (5.6) with the appropriate diffusion coefficient (5.12), either by solving directly Eqs. (4.20) and (5.7) and making use of the result (5.11), or via the associated continuity equation.

The diffusion equations (5.2) and (5.11) suffice to deter-
mine all macroscopic properties of the system, in the same sense that the Landau-Boltzmann equation for quasiparticles in clean\textsuperscript{16} or weakly disordered\textsuperscript{17} systems embodies all properties of a Fermi liquid. We shall return to discussing the diffusion equation in a following paper\textsuperscript{12} that deals with interacting disordered systems.

VI. CONCLUDING REMARKS

In this paper we have provided a systematic derivation of the Boltzmann-type transport equations that hold in the hydrodynamic regime for noninteracting electrons in the presence of strong disorder. We have approached the problem by relying on the ordinary diagrammatic theory, thus focusing mainly in the difficulties one encounters when trying to disentangle the functional form of the collision integral of the Boltzmann equation from the diagrammatic structure. Our approach makes manifest both virtues and limitations of a search for a kinetic equation for a phase-space distribution function, when quantum-interference effects need to be taken into account. Nonetheless, we have been able to extract physically meaningful expressions for the effective scattering kernel of the Boltzmann equation in the presence of nonmagnetic and magnetic impurities, thereby treating the two cases on equal footing and providing an understanding at a microscopic level of the processes leading to electronic localization when time-reversal invariance is lacking. We have also discussed how our findings are consistent with general symmetry requirements that could in turn be used to limit the possible form of the scattering kernel for alternative universality classes. The need for introducing a spatial nonlocality in the scattering kernel in the presence of strongly disordered magnetic impurities is actually a novel feature of our analysis, which has been hindered in the cases treated so far of weak disorder and of strongly disordered nonmagnetic impurities only.

There remains to show how and to what extent this kind of approach to the kinetic equation can be carried over to the case of interacting electrons in the presence of strong disorder. We postpone this subject to a following paper.\textsuperscript{12}

APPENDIX A: CALCULATION OF THE DISTRIBUTION FUNCTION

BY LINEAR-RESPONSE TECHNIQUES

Quite generally, the Fourier transform \( \mathcal{L} \) of the two-particle correlation function (2.8) satisfies a Bethe-Salpeter equation of the form

\[
\mathcal{L} \left[ k + \frac{Q}{2}, \omega + \frac{\Omega}{2} - k + \frac{Q}{2}, -\omega + \frac{\Omega}{2} \right] = G \left[ k + \frac{Q}{2}, \omega + \frac{\Omega}{2} \right] G \left[ k - \frac{Q}{2}, -\omega - \frac{\Omega}{2} \right] \\
\times \left[ 1 + \int \frac{dk'}{(2\pi)^d} \int \frac{d\omega'}{2\pi} \Xi_{s}(k\omega, k'\omega'; Q\Omega) \mathcal{L} \left[ k' + \frac{Q}{2}, \omega' + \frac{\Omega}{2} - k' + \frac{Q}{2}, -\omega' + \frac{\Omega}{2} \right] \right],
\]

(A1)

where \( G \) is the ordinary (impurity-averaged) single-particle Green's function and \( \Xi_{s} \) stands for the singlet component of the irreducible two-particle kernel. Equation (A1) is depicted graphically in Fig. 5. In the absence of the electron-electron interaction the integration over \( \omega' \) on the right-hand side of Eq. (A1) is missing. In addition, the kernel \( \Xi_{s} \) can be taken to be independent of \( \omega \) for small values of the external momentum \( Q \) and frequency \( \Omega \), and the single-particle Green's function is given by

\[
G_{\pm}(q, \nu) = \frac{1}{\nu - \epsilon_{q}^{\pm} - i \frac{\tau}{2}}.
\]

(A2)

Here the + and - signs correspond to positive and negative values of the frequency \( \nu \), respectively, and \( \tau \) is the total impurity scattering time in the Born approximation.\textsuperscript{20}

The procedure adopted in Sec. II to define the phase-space distribution function requires us to integrate a pair of single-particle Green's functions [cf. Eq. (A1)] over their common kinetic energy \( \epsilon_{k} \). Reckoning the zero of the single-particle energies at the chemical potential \( \mu \), we obtain the following for the various cases (\( \Omega > 0 \)).\textsuperscript{17}

(i) For \( \Omega/2 < |\omega| << \mu \),

\[
\int_{-\mu}^{+\infty} \frac{d\epsilon_{k}}{2\pi} G_{\pm} \left( k + \frac{Q}{2}, \omega + \frac{\Omega}{2} \right) \approx -\frac{1}{2\pi \mu}.
\]

(A3)

(ii) For \( |\omega| < \Omega/2 \),

\[
\int_{-\mu}^{+\infty} \frac{d\epsilon_{k}}{2\pi} G_{\pm} \left( k + \frac{Q}{2}, \omega + \frac{\Omega}{2} \right) \approx \frac{1}{\tau + i\omega_{k}Q - \Omega}.
\]

(A4)
to leading order in the small parameter \((\tau \mu)^{-1} (v_F = k_F / m\) being the Fermi velocity). Following a common practice in localization problems, we shall consistently take the limit of the above expressions for \(\mu \approx \infty\) whereby the right-hand side of Eq. (A3) vanishes.

**APPENDIX B: CALCULATION OF CERTAIN ANGULAR AVERAGES**

Equation (4.12) of the text gets generalized to the product of \(n/2\) pairs of scalar products \(\mathbf{a}_i \cdot \mathbf{q} (i = 1, 2, \ldots, n)\) in the following way:

\[
\left\langle \langle \left( \mathbf{a}_1 \cdot \mathbf{q} \right) \cdots \left( \mathbf{a}_n \cdot \mathbf{q} \right) \right\rangle = \left\langle \langle \mathbf{q} \rangle \right\rangle \prod_{s=1}^{n/2} \left[ d + (n - 2s) \right]^{-1} \left( \sum_C \left( \mathbf{a}_1 \cdot \mathbf{a}_2 \right) \cdots \left( \mathbf{a}_{n-1} \cdot \mathbf{a}_n \right) \right),
\]

where the sum extends over all pairwise contractions of the \(n\) vectors \(\mathbf{a}_i (n \text{ even})\). In this expression \(\left\langle \langle \cdots \right\rangle\) denotes the average over the angular variables of the vector \(\mathbf{q}\), which may contain in its measure a weight function invariant under the transformation \(\mathbf{q} \to \mathcal{R} \mathbf{q}\) [where \(\mathcal{R}\) stands for an element of the orthogonal group \(O(d)\) in \(d\) dimensions].

Equation (B1) has been inferred upon working out explicitly the cases \(n = 2, 4, 6\) which are needed in the calculation of Eq. (4.9). Notice that the result [(B1)] is properly invariant under common rotations of all vectors \(\mathbf{a}_i\), and that in one dimension the factor in the denominator correctly reduces to the number of pairwise contractions of \(n\) objects:

\[
\prod_{s=1}^{n/2} \left[ 1 + (n - 2s) \right] = (n - 1)!!
\]

[when \(d = 1\) all terms of the sum on the right-hand side of Eq. (B1) are in fact equal].

Although Eq. (B1) holds for any (positive) integer dimension \(d\) for which the transformations \(\mathcal{R}\) of the orthogonal group \(O(d)\) are meaningful, we shall extend its validity by analytically continuing at intermediate dimensions in the spirit of the \(\varepsilon\) expansion.

From Eq. (B1) we obtain in particular

\[
\left\langle \left\langle (Q \cdot q) (\hat{k} \cdot q) \right\rangle \right\rangle = \frac{\left\langle \langle q^4 \rangle \right\rangle}{d(d+2)} [Q^2 + 2(Q \cdot \hat{k})^2],
\]

\[
\left\langle \left\langle (Q \cdot q)(\hat{k} \cdot q)^2(\hat{k} \cdot q) \right\rangle \right\rangle = \frac{\left\langle \langle q^4 \rangle \right\rangle}{d(d+2)} [(Q \cdot \hat{k})^2 + 2(Q \cdot \hat{k})(\hat{k} \cdot \hat{k})],
\]

\[
\left\langle \left\langle (Q \cdot q)(\hat{k} \cdot q)(\hat{k} \cdot q)^2 \right\rangle \right\rangle = \frac{\left\langle \langle q^6 \rangle \right\rangle}{d(d+2)(d+4)} [Q^2 + 2Q \cdot \hat{k} \cdot \hat{k}^2 + 2(Q \cdot \hat{k})^2 + 2(Q \cdot \hat{k})^2 + 8(Q \cdot \hat{k})(Q \cdot \hat{k})(\hat{k} \cdot \hat{k})],
\]

which are relevant for the calculation of Eq. (4.9) of the text.

**APPENDIX C: EXTRACTION OF THE INFRARED DIVERGENCE OF CERTAIN INTEGRALS OCCURRING AT TWO-LOOP ORDER**

The following are the complete expressions of the functions \(\gamma_i(\Omega; \hat{k} \cdot \hat{k}')\) occurring in the integrand of Eq. (4.13) of the text:

\[
\gamma_1(\Omega; \hat{k} \cdot \hat{k}') = -4(\hat{k} \cdot \hat{k}') J_2(\Omega) + \frac{18}{d+2} [d(\hat{k} \cdot \hat{k}')^2 - 1] J_6(\Omega),
\]

\[
\gamma_2(\Omega; \hat{k} \cdot \hat{k}') = \frac{10}{d} [1 - d(\hat{k} \cdot \hat{k}')^2] J_2(\Omega) + 2 J_3(\Omega) - \frac{4}{d} J_5(\Omega) - \frac{24}{d+2} [1 - d(\hat{k} \cdot \hat{k}')] J_6(\Omega)
\]

\[
+ \frac{18}{d+2} [d(\hat{k} \cdot \hat{k}')^2 - 1] J_6(\Omega),
\]

\[
\gamma_3(\Omega; \hat{k} \cdot \hat{k}') = [-25 + 6d(\hat{k} \cdot \hat{k}')] J_2(\Omega) - 2d J_3(\Omega) + 10 J_5(\Omega) + \frac{24}{d+2} [5 - d(\hat{k} \cdot \hat{k}')] J_6(\Omega)
\]

\[
- \frac{24}{d+2} J_6(\Omega) - \frac{288}{(d+2)(d+4)} J_{12}(\Omega),
\]

\[
\gamma_4(\Omega; \hat{k} \cdot \hat{k}') = [-20 + 25d(\hat{k} \cdot \hat{k}')] J_2(\Omega) - 4d J_3(\Omega) + 8 J_5(\Omega) + \frac{24}{d+2} [2 - 5d(\hat{k} \cdot \hat{k}')] J_6(\Omega)
\]

\[
- \frac{18d}{d+2} [d(\hat{k} \cdot \hat{k}')^2 - 1] J_6(\Omega) - 4d J_{10}(\Omega) + 16 J_{11}(\Omega) - \frac{32}{d+2} J_{13}(\Omega) + \frac{288d}{(d+2)(d+4)} (\hat{k} \cdot \hat{k}') J_{12}(\Omega),
\]
\[
\gamma(\Omega; \hat{k}, \hat{k}') = \frac{25}{d} J_5(\Omega) + 4 J_3(\Omega) - J_4(\Omega) - \frac{20}{d} J_2(\Omega) - \frac{4}{d+2} \left[ \frac{120}{d} + 18(d(\hat{k} \cdot \hat{k}')^2 - 1) \right] J_6(\Omega) \\
+ \frac{4}{d+2} J_7(\Omega) + \frac{48}{d(d+2)} J_8(\Omega) + 4(\hat{k} \cdot \hat{k}') J_{11}(\Omega) \\
+ \frac{32}{(d+2)(d+4)} \left[ \frac{4}{d} + [d(\hat{k} \cdot \hat{k}')^2 - 1] \right] J_{12}(\Omega) - \frac{16}{d+2}(\hat{k} \cdot \hat{k}') J_{13}(\Omega). \tag{C5}
\]

Here the two-loop integrals \( J_i(\Omega) \) (\( i = 1, 2, \ldots, 13 \)) are defined by

\[
J_1(\Omega) = \tau \int \frac{d q_1}{(2\pi)^d} \int \frac{d q_2}{(2\pi)^d} \left( q_1^2 + q_2^2 \right) \left( q_1^2 + q_2^2 - 4i \Omega \right) \\
J_2(\Omega) = \int \frac{d q_1}{(2\pi)^d} \int \frac{d q_2}{(2\pi)^d} \left( q_1^2 - i \Omega \right) \left( q_2^2 - i \Omega \right) \left( (q_1 + q_2)^2 - i \Omega \right) \\
J_3(\Omega) = \int \frac{d q_1}{(2\pi)^d} \int \frac{d q_2}{(2\pi)^d} \left( q_1^2 + q_2^2 \right) \\
J_4(\Omega) = \int \frac{d q_1}{(2\pi)^d} \int \frac{d q_2}{(2\pi)^d} \left( q_1^2 - i \Omega \right) \left( q_2^2 - i \Omega \right) \left( (q_1 + q_2)^2 - i \Omega \right) \\
J_5(\Omega) = \int \frac{d q_1}{(2\pi)^d} \int \frac{d q_2}{(2\pi)^d} \left( q_1^2 + q_2^2 \right) \\
J_6(\Omega) = \int \frac{d q_1}{(2\pi)^d} \int \frac{d q_2}{(2\pi)^d} \left( q_1^2 + q_2^2 \right) \left( q_1 + q_2 \right) \left( q_1 + q_2 \right) \\
J_7(\Omega) = \int \frac{d q_1}{(2\pi)^d} \int \frac{d q_2}{(2\pi)^d} \left( q_1^2 - i \Omega \right) \left( q_2^2 - i \Omega \right) \left( (q_1 + q_2)^2 - i \Omega \right) \\
J_8(\Omega) = \int \frac{d q_1}{(2\pi)^d} \int \frac{d q_2}{(2\pi)^d} \left( q_1^2 + q_2^2 \right) \\
J_9(\Omega) = \tau \int \frac{d q_1}{(2\pi)^d} \int \frac{d q_2}{(2\pi)^d} \left( q_1^2 + q_2^2 \right) \\
J_{10}(\Omega) = \frac{1}{\tau} \int \frac{d q_1}{(2\pi)^d} \int \frac{d q_2}{(2\pi)^d} \left( q_1^2 - i \Omega \right) \left( q_2^2 - i \Omega \right) \left( (q_1 + q_2)^2 - i \Omega \right) \\
J_{11}(\Omega) = \frac{1}{\tau} \int \frac{d q_1}{(2\pi)^d} \int \frac{d q_2}{(2\pi)^d} \left( q_1 + q_2 \right)^2 \\
J_{12}(\Omega) = \frac{1}{\tau} \int \frac{d q_1}{(2\pi)^d} \int \frac{d q_2}{(2\pi)^d} \left( q_1 + q_2 \right)^6 \\
J_{13}(\Omega) = \frac{1}{\tau} \int \frac{d q_1}{(2\pi)^d} \int \frac{d q_2}{(2\pi)^d} \left( q_1 + q_2 \right)^4. \tag{C6}
\]

Notice that all the integrals (C6) have physical dimensions of a frequency to the power \( d - 2 \).

From what we have argued in Sec. IV, only the knowledge of the (infrared) singular part of the integrals \( J_1(\Omega) + J_9(\Omega) \) is required to obtain the effective scattering kernel of the Boltzmann equation in the hydrodynamic regime. We then concentrate on this task in the following.

From power counting the integral \( J_1(\Omega) \) is expected to contain a term proportional to \( \Lambda^d \ln(\Omega/D_0 \Lambda^2) \) which needs to be subtracted off, \( \Lambda \) being an ultraviolet cutoff. To this end, it is convenient to consider the following \( Q \)- and \( \Omega \)-dependent integral

\[
\mathcal{H}(Q, \Omega) = \tau \int \frac{d q_1}{(2\pi)^d} \int \frac{d q_2}{(2\pi)^d} \left( q_1^2 + q_2^2 \right) \\
\left( q_1^2 - i \Omega \right) \left( q_2^2 - i \Omega \right) \left( (q_1 + q_2)^2 - i \Omega \right) \left( (q_1 + q_2 + Q)^2 - i \Omega \right), \tag{C7}
\]

whose expansion in powers of \( Q \) gives
\[ \mathcal{H}(Q, \Omega) = J_1(\Omega) + 4i \Omega \tau J_3(\Omega) + \tau Q^2 \left[ \frac{4}{d} J_7(\Omega) - J_4(\Omega) \right] + \cdots . \]  \hfill (C8)

[Notice that \( J_4(\Omega) \) and \( J_7(\Omega) \) enter the expression (C5) in this particular combination.] The integrand of Eq. (C7) can then be manipulated to give

\[ \mathcal{H}(Q, \Omega) = \tau \int \frac{d q_1}{(2\pi)^d} \int \frac{d q_2}{(2\pi)^d} \left[ \phi(q_1, q_2; Q, \Omega) + \frac{(Q^2 + 5i \Omega)(q_1^2 - i \Omega)(q_2^2 - i \Omega) - 4\Omega^2}{(q_1^2 - i \Omega)(q_2^2 - i \Omega)(q_1 + q_2 + Q)^2 - i \Omega} \right] , \]  \hfill (C9)

where the symmetric function

\[ \phi(q_1, q_2; Q, \Omega) = \frac{1}{q_1^2 - i \Omega} + \frac{1}{q_2^2 - i \Omega} + \frac{1}{(q_1 + q_2 + Q)^2 - i \Omega} - \frac{2}{(q_1 + q_2 + Q)^2 - i \Omega} \left( \frac{q_1}{q_1^2 - i \Omega} + \frac{q_2}{q_2^2 - i \Omega} \right) \]  \hfill (C10)

is suitably devised to yield alone the \( \Lambda^4 \ln(\Omega/D_0 \Lambda^2) \) term.

A comment on the way the cutoff \( \Lambda \) has been handled is in order at this point. The occurrence of the combination \( q_1 + q_2 + Q \) in the argument of a direct ladder in Eq. (C10) makes it convenient to regularize each direct ladder therein by the Schwinger parametrization\textsuperscript{27}

\[ \frac{1}{q^2 - i \Omega} \to \int_0^\infty d \alpha \exp \{- \alpha(q^2 - i \Omega)\} , \]  \hfill (C11)

where \( q \) is now allowed to range over the whole momentum space. In this way one verifies that the last two terms on the right-hand side of Eq. (C10) give a vanishing contribution upon integration over \( q_1 \) and \( q_2 \), while the remaining terms produce the desired \( \Lambda^4 \ln(\Omega/D_0 \Lambda^2) \) dependence. In the second term within the large parentheses on the right-hand side of Eq. (C9), on the other hand, the \( Q \) occurring in the denominator can be disregarded, and the integrals over \( q_1 \) and \( q_2 \) can be conveniently regularized by dimensional regularization\textsuperscript{27} (whereby the ultraviolet cutoff is disposed of by extrapolating the integration to noninteger dimensions). We then obtain

\[ \mathcal{H}(Q, \Omega) = 2\tau (Q^2 + 5i \Omega) I_3^2(\Omega) , \]  \hfill (C12)

where now

\[ I_d(\Omega) = \int \frac{d q}{(2\pi)^d} \frac{1}{q^2 - i \Omega} = \frac{\Gamma(1 - d/2)}{(2\sqrt{\pi})^d} (-i \Omega)^{d/2 - 1} \approx \frac{(-i \Omega)^{d/2}}{2\pi \varepsilon} \]  \hfill (C13)

near two dimensions within the \( \varepsilon(d - 2) \) expansion (\( \Gamma \) being the Euler's gamma function). Notice that the term proportional to \( \Omega^2 \) on the right-hand side of Eq. (C9) has been disregarded to arrive to Eq. (C13), since it gives only a finite contribution near two dimensions. Comparison of Eq. (C13) with Eq. (C8) yields eventually

\[ J_1(\Omega) = 2i \Omega \tau [5I_3^2(\Omega) - 2J_3(\Omega)] , \]  \hfill (C14)

\[ \frac{4}{d} J_7(\Omega) - J_4(\Omega) = 2I_3^2(\Omega) . \]  \hfill (C15)

The infrared divergence of \( J_3(\Omega) \) can be extracted directly from its definition by adopting dimensional regularization and by disregarding a term that remains finite in two dimensions. With these provisions one obtains

\[ J_3(\Omega) = 2I_3^2(\Omega) , \]  \hfill (C16)

which gives

\[ J_1(\Omega) = 2i \Omega \tau I_3^2(\Omega) . \]  \hfill (C17)

The infrared divergence of the remaining integrals can be extracted in a similar way. One obtains

\[ J_2(\Omega) = I_2^2(\Omega) , \]  

\[ J_5(\Omega) = 2I_3^2(\Omega) - 2\frac{\Gamma(2 - d/2)}{(2\sqrt{\pi})^d} (-i \Omega)^{d/2 - 1} I_d(\Omega) = dI_3^2(\Omega) , \]  

\[ J_6(\Omega) = 2I_2^2(\Omega) , \]  

\[ J_4(\Omega) = 2I_3^2(\Omega) + \frac{2}{(2\sqrt{\pi})^d} \left[ \frac{1}{2} \Gamma(3 - d/2) - 2\Gamma(2 - d/2) \right] (-i \Omega)^{d/2 - 1} I_d(\Omega) = \frac{d}{2}(d + 2)I_3^2(\Omega) , \]  \hfill (C18)

where use has been made of Eq. (C13) and of the recurrence property of the gamma function. By the same token, \( J_5(\Omega) \) can safely be dropped, being harmless in the infrared regime near two dimensions.


\textsuperscript{3}See, e.g., \textit{Anderson Localization}, edited by Y. Nagaoka and H. Fukuyama (Springer, New York, 1982).
8Generalizations of the semiclassical Boltzmann equation to include quantum effects have also been considered in different contexts, such as the electronic transport in nondegenerate semiconductors: L. Reggiani, P. Lugli, and A. P. Jauho, Phys. Rev. B 36, 6602 (1987).
13G. Strinati, C. Castellani, and C. Di Castro, Phys. Rev. B 39, 4824 (1989). In this reference we have defined the distribution function in terms of the Keldish function. At leading order in the external frequency and wave vector we consider throughout, the resulting distribution function coincides with the one defined in the present paper.
14We set $\hbar = 1$ throughout.
15Compare, e.g., L. E. Reichl, A Modern Course in Statistical Physics (Edward Arnold, London, 1980), Sec. 7-G.
18Compare, e.g., Sec. 6-2 of the first part of Ref. 10.
21The distinction we have made between Eq. (3.7) for the full distribution function and Eq. (3.5) for its dynamic part uncovers the difference between two distinct kinetic equations [Eqs. (19) and (13), respectively] of a paper by W. L. McMillan [Phys. Rev. B 31, 2750 (1985)], which will be discussed further in a following paper dealing with the interacting case.
23In the context of the nonlinear $\sigma$ model, into which the theory of localization can be mapped, the terms proportional to $A^4$ are actually canceled by the contributions from the integration measure [see E. Brézin, S. Hikami, and J. Zinn-Justin, Nucl. Phys. B 165, 528 (1980); see also the general Ref. 3]. We shall assume that this cancellation occurs also in the present context, without searching explicitly for the effective counterterms of our perturbative scheme, since they will appear anyway as subleading contributions as far as the infrared divergencies are concerned.
24The term $-i\Omega$ within brackets at the right-hand side of the Boltzmann equations (3.12) and (4.22) is understood to multiply only the linear part $\delta n$ of the distribution function.
25One usually assumes the magnetic scattering time $\tau_m$ to be much longer than its nonmagnetic counterpart. Notice also that the impurity-averaged single-particle Green’s function remains diagonal in the spin indices in the presence of magnetic impurities.
26Compare, e.g., K. Huang, Statistical Mechanics (Wiley, New York, 1963), Sec. 4.1.
28The appearance of the integrals $J_{10}$, $J_{11}$, and $J_{13}$ [that have stronger than $(\log)^3$ singularities in two dimensions] in the functions $\gamma_i$, is due to the form in which Eq. (4.13) has been cast in. In fact, when the dependence on $\vec{k}'$ of the function $\mathcal{L}^{(1)}$ on the right-hand side of Eq. (4.13) is made explicit and the integral over $\vec{k}'$ is evaluated, the coefficients that multiply the integrals $J_{10}$, $J_{11}$, and $J_{13}$ vanish identically by inversion symmetry.
29Similar expressions have been considered at two-loop order by E. Brézin and J. Zinn-Justin, Phys. Rev. B 14, 3110 (1976).